GRAVITATION 6

The phenomenon of mutual attraction between any two bodies is known as Gravitation.

ACCELERATION OF THE MOON:

The centripetal acceleration of the moon can be determined by assuming that the moon revolves around the earth in circular orbit. Newton calculated the centripetal acceleration of the moon by applying Huygen's formula for centripetal acceleration i.e.

$$\vec{a}_c = \frac{-v^2}{r}\hat{r}$$

Where $\hat{\mathbf{r}}$ = unit vector directed towards centre of the moving body.

v = speed of the moving body

r = radius of the orbit or circle.

Negative sign shows that a_c is directed towards the centre of the circle.

The magnitude of the centripetal acceleration is given by

$$a_c = \frac{v^2}{r}$$

$$v = r\omega$$

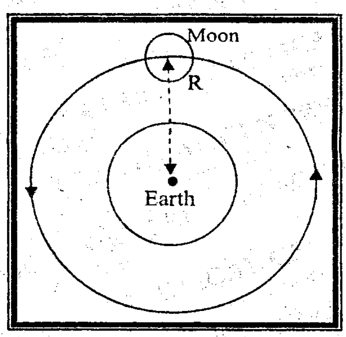
$$\therefore a_c = \frac{r^2\omega^2}{r} = r\omega^2$$

$$\text{As } \omega = \frac{\theta}{t}$$

For one rotation $\theta = 2\pi$ and $t = T$ = time period

$$\therefore \omega = \frac{2\pi}{T}$$

$$a_c = \frac{4\pi^2 r}{T^2} \text{ ----------(1)}$$



If the distance of the moon from the centre of the earth is "R" and acceleration of the moon is "a_m" then eq. (1) becomes.

$$a_m = \frac{4\pi^2 R}{T^2}$$

As $R = 3.84 \times 10^8$ m

And $T = 27.3$ days $= 27.3 \times 24 \times 60 \times 60$

$= 2.36 \times 10^6$ sec

$$\therefore a_m = \frac{4 \times (3.142)^2 \times 3.84 \times 10^8}{(2.36 \times 10^6)2}$$

$$a_m = 2.72 \times 10^{-3}\ m/S^2$$

APPROACH TO THE LAW OF UNIVERSAL GRAVITATION:

(1) Newton arrived at the law of universal gravitation by comparing the acceleration of the moon and the acceleration of a body falling freely near the surface of the earth.

The acceleration of the moon = $a_m = 2.72 \times 10^{-3} \frac{m}{s^2}$ --------- (1)

The acceleration of a body falling freely near the surface of the earth $= g = 9.8 \frac{m}{s^2}$ ---------(2) dividing eq (2) by eq (1)

$$\frac{g}{a_m} = \frac{9.8}{2.72 \times 10^{-3}} = \frac{3603}{1} = \frac{(60)^2}{1}$$

$$\frac{a_m}{g} = \frac{1}{(60)^2} \text{-------------(3)}$$

The distance of the moon from the centre of the earth $= R = 3.84 \times 10^8 m$

$R^2 = 14.7 \times 10^{16}\ m^2$

The distance of the body falling freely near the surface of the earth

$Re = 6.38 \times 10^6 m$ --------------- (4)

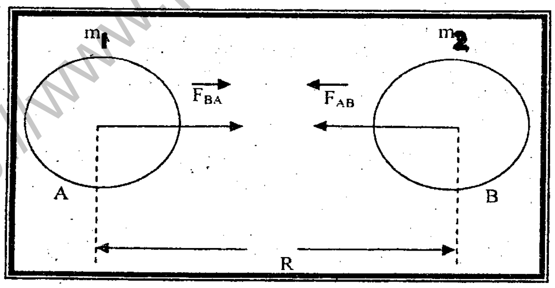
$Re^2 = 40.7 \times 10^{12}\ m^2$ ------------ (5)

Dividing eq (4) by (5)

$$\frac{R^2}{Re^2} = \frac{14.7 \times 10^{16}}{40.7 \times 10^{12}} = \frac{3611.8}{1} = \frac{(60.2)^2}{1}$$

OR

$$\frac{Re^2}{R^2} = \frac{1}{(60)^2} \text{ -------------- (6)}$$



Comparing eq (3) and (6)

$$\frac{a_m}{g} = \frac{Re^2}{R^2}$$

This shows that acceleration is inversely proportional to the square of the distance.

By Newton's second law of motion acceleration is directly proportional to the force i.e.

$a_m \propto F$

$\therefore$ $\boxed{F \propto \frac{1}{R^2}}$

(2) Consider two bodies A and B of masses "m_1" and "m_2" at a distance "R" from each other. The force exerted by A on body B "F_{AB}" directly depends upon mass m_B of the body B i.e.

$F_{AB} \propto m_B$

Similarly the force exerted by body B on body A "F_{BA}" directly depends upon mass m_1 of body A i.e. $F_{BA} \propto m_1$.

As the forces F_{AB} and F_{BA} are mutual attractions and are equal in magnitude.

$\therefore F_{AB} = F_{BA} = F$ (say)

Thus $F \propto m_2$

$F \propto m_1$

$\Rightarrow$ $\boxed{F \propto m_1 m_2}$

From the above relations we can state the law of gravitation.

NEWTON'S LAW OF UNIVERSAL GRAVITATION:

Statement:

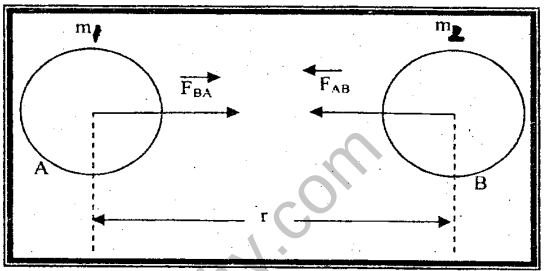
"Every body in the universe attracts every other body with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centres of gravity."

Explanation:

Consider two bodies A and B of masses "m_1" and "m_2" at a distance "r" from each other. If "F_{AB}" be the magnitude of the force exerted by the body A on B then by this law,

$F_{AB} \propto m_1 m_2$ and

$F_{AB} \propto \frac{1}{r^2}$



By combining above relations

$$F_{AB} \propto \frac{m_1 m_2}{r^2}$$

$$\boxed{F_{AB} = G\frac{m_1 m_2}{r^2}}$$

Where G = Universal Gravitational constant and its value in S.I. units is $6.673 \times 10^{-11}\ \frac{Nm^2}{kg^2}$

Gravitation force in vector form is given by

$$\vec{F}_{AB} = \frac{-Gm_1 m_2}{r^2}\hat{r}_{BA}$$

Where $\hat{r}_{BA}$ = unit vector directed from 'B' to 'A'. Negative sign shows that the force is attractive. By Newton's third law of motion body 'B' will also exert some force on body 'A' and is given by

$$\vec{F}_{BA} = \frac{-Gm_1 m_2}{r^2}\hat{r}_{AB}$$

Where $\hat{r}_{AB}$ = unit vector directed from A to B

Mass of the Earth:

Consider a body of mass m placed at the surface of the earth. Let Me be the mass of the earth and Re be its radius. By the law of universal gravitation, force between the body and the earth is given by

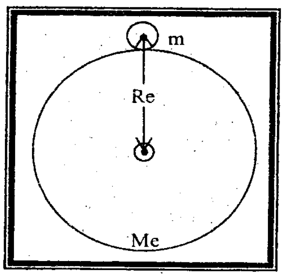
$$F = G\frac{mMe}{Re^2} \text{ ---------- (1)}$$

Where

G = gravitational constant.

The force with which earth attracts a body towards its centre is called weight of the body i.e.

$$F = mg \text{ ---------- (2)}$$



Comparing eq. (1) and eq.(2)

$$\frac{GmMe}{Re^2} = mg$$

$$\boxed{g = \frac{GMe}{Re^2}}$$

$$Me = \frac{gRe^2}{G}$$

Since $g = 9.8\ m^2/S^2$

$G = 6.673 \times 10^{-11}\ N.m^2/Kg^2$

$Re = 6.38 \times 10^6 m$

$$\therefore Me = \frac{9.8x(6.38x10^6)^2}{6.673x10^{-11}}$$

$$Me = 5.98 \times 10^{24}\ kg$$

Average Density of the Earth:

Mass per unit volume of a body is called its density. If Me be the mass of the earth and Ve be the volume, then its average density ρ is given by

$$\rho = \frac{Me}{Ve}$$

As earth is considered to be spherical then.

$$Ve = \frac{4}{3}\pi R_e^3$$

Thus $\rho = \dfrac{Me}{\frac{4}{3}\pi R_e^3}$

Since $Me = 5.98 \times 10^{24}$ Kg

$Re = 6.38 \times 10^6 m$

$\therefore\ \pi = 3.142$

$$\rho = \frac{5.98x10^{24}}{\frac{4}{3}x3.142x(6.38x10^6)^3}$$

$$\rho = 5.52 \times 10^3\ \frac{Kg}{m^3}$$

Mass of the Sun:

Let Ms be the mass of the sun and Me be the mass of the earth orbiting around the sun. If R be the radius of the orbit of the earth, then gravitational force between the earth and the sun is given by.

$$F = \frac{GMsMe}{R^2} \text{ ------------- (1)}$$

The centripetal acceleration of the earth is given by

$$a_c = \frac{4\pi^2 R}{T^2}$$

where

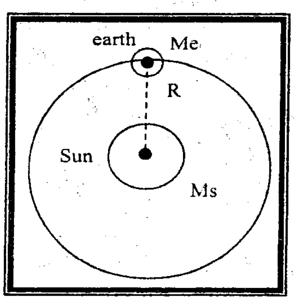
T = period of revolution of the earth around the sun.

From Newton's second law of motion

$F = Me\ a_c$

$$F = Me\ \frac{4\pi^2 R}{T^2}$$

$$F = \frac{4\pi^2\ MeR}{T^2} \longrightarrow (2)$$



Comparing eq (1) and (2)

$$\frac{GMsMe}{R^2} = \frac{4\pi^2 MeR}{T^2}$$

$$\boxed{Ms = \frac{4\pi^2 R^3}{GT^2}}$$

Since $\pi = 3.142$

$R = 1.49 \times 10^{11}$ m

$G = 6.673 \times 10^{-11} \frac{Nm^2}{Kg^2}$

$T = 365.30$ days

$= 365.30 \times 24 \times 60 \times 60 = 31.56 \times 10^6$ seconds

Then $Ms = \frac{4 \times (3.142)^2 \times (1.49 \times 10^{11})^3}{6.673 \times 10^{-11} \times (31.56 \times 10^6)^2}$

$$\boxed{Ms = 1.99 \times 10^{30} \text{ Kg}}$$

VARIATION OF "g" WITH ALTITUDE:

The acceleration due to gravity at the surface of the earth is given by

$$g = \frac{GMe}{Re^2} \text{ --------- (1)}$$

Where G = Gravitational constant

Me = Mass of the earth

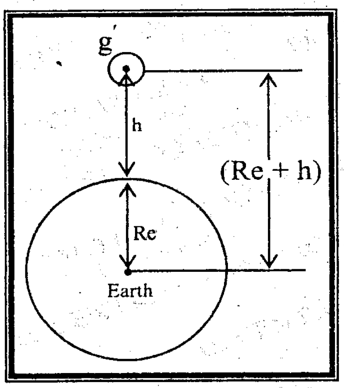
Re = Radius of the earth

G and Me are constant therefore g varies with the distance from the centre of the earth.

If g' be the acceleration due to gravity at distance (Re + h) from the centre of earth then it is given by

$$g' = \frac{GMe}{(Re+h)^2} \text{ -------- (2)}$$

Dividing eq. (2) by eq. (1)



$$\frac{g'}{g} = \frac{\frac{GMe}{(Re+h)^2}}{\frac{GMe}{Re^2}} = \frac{GMe}{(Re+h)^2} \times \frac{Re^2}{GMe} = \frac{Re^2}{(Re+h)^2}$$

$$\frac{g'}{g} = \frac{Re^2}{Re^2\left[1+\frac{h}{Re}\right]^2} = \frac{1}{\left[1+\frac{h}{Re}\right]^2}$$

$$\frac{g'}{g} = \left[1+\frac{h}{Re}\right]^{-2}$$

Using binomial theorem

$$\frac{g'}{g} = 1 + \frac{(-2)}{1!}\left[\frac{h}{Re}\right] + \frac{(-2)(-3)}{2!}\left[\frac{h}{Re}\right]^2 + \text{--------}$$

$$= 1 - 2\left[\frac{h}{Re}\right] + \frac{6}{2 \times 1}\left[\frac{h}{Re}\right]^2 + \text{---}$$

$$= 1 - 2\frac{h}{Re} + 3\frac{h^2}{Re^2} + - - - - -$$

If h << Re then we can neglect terms containing higher power of $\frac{h}{R_e}$

$$\therefore \frac{g'}{g} = 1 - \frac{2h}{Re}$$

$$\therefore g' = g(1 - \frac{2h}{Re}) \Rightarrow g' = g - \frac{2h}{Re} \Rightarrow g' - g = \frac{2gh}{Re}$$

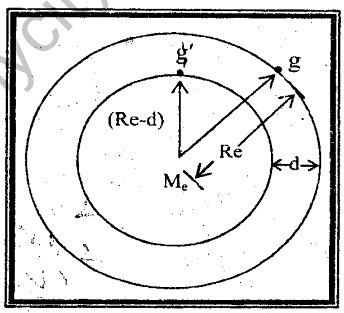
$$\boxed{\Delta g = -\frac{2gh}{Re}}$$

This shows that acceleration due to gravity decreases with altitude.

VARIATION OF "g" WITH DEPTH:

The acceleration due to gravity at the surface of earth is given by

$$g = \frac{GMe}{Re^2}$$



Where

G = Gravitational constant.
Re = radius of the earth
Me = Mass of the earth

$$\because \text{Density} = \frac{\text{Mass}}{\text{Volume}}$$

$$\rho = \frac{Me}{Ve}$$

As earth is considered to be spherical then

$$Ve = \frac{4}{3}\pi Re^3$$

$$Me = V_e \rho$$

$$Me = \frac{4}{3}\pi Re^3 \rho$$

$$\therefore g = \frac{G\frac{4}{3}\pi Re^3 \rho}{Re^2}$$

$$g = G\frac{4}{3}\pi Re \ \rho$$

$$g = \frac{4}{3}\pi G \rho Re \text{ --------- (1)}$$

where

ρ = Average density of the earth

If g' be the acceleration due to gravity at depth "d" below the surface of the earth then

$$g' = \frac{GM'e}{(Re-d)^2}$$

Where M'_e = effective mass of the earth for attraction at depth "d" and is given by

$$M'e = \frac{4}{3}\pi(Re-d)^3\rho$$

$$\therefore g' = \frac{G\frac{4}{3}\pi(Re-d)^3\rho}{(Re-d)^2}$$

$$g' = 4/3\ \pi G\rho(Re-d) \text{ -------- } (2)$$

Dividing eq (2) by eq (1)

$$\frac{g'}{g} = \frac{\frac{4}{3}\pi G\rho(Re-d)}{\frac{4}{3}\pi G\rho Re} = \frac{Re-d}{Re} = \frac{Re}{Re} - \frac{d}{Re}$$

$$\frac{g'}{g} = 1 - \frac{d}{Re}$$

$$g' = g(1 - \frac{d}{Re})$$

$$\boxed{g' = g - \frac{gd}{Re}}\quad g' - g = -\frac{d}{Re}g \Rightarrow \boxed{\Delta g = -\frac{d}{Re}g}$$

This shows that acceleration due to gravity decreases by a factor $\frac{gd}{Re}$ as we go below the surface of the earth.

At the Centre of the Earth:

If d = Re then eq (3) becomes

$$g' = g - g\frac{Re}{Re}$$

$$g' = g - g$$

$$\boxed{g' = 0}$$

This shows that the acceleration due to gravity becomes zero at the centre of the earth.

Weight:

When a force is applied on a body to keep it at rest or to prevent it from accelerating in a frame of reference then its equal and opposite force is called "weight"

If other forces are negligible then force of attraction of the earth on the body is known as weight of the body.

The weight of a body can be found by Newton's second law of motion i.e. F = ma

For freely-falling bodies a = g and F = W

$$\therefore \boxed{W = mg}$$

Weight of the body is not a fixed quantity. It depends upon the location as well as on the motion of the frame of reference. Weight of a body is slightly less than the force of attraction of the earth. It is due to the rotation of the earth and buoyancy of atmosphere.

Weight of the body is maximum at the poles.

WEIGHTLESSNESS IN SATELLITES:

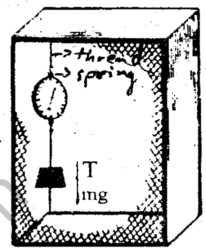
1) To understand weightlessness, consider a body of mass "m" suspended from a spring balance by means of a thread attached to the ceiling of an elevator.

There are two forces acting on the body.

(i) Force of gravity = mg **(ii)** Tension in the string = $T = F_w$

(a) When the elevator at rest or moving with uniform velocity:

When the elevator at rest or moving with uniform velocity then $a = 0$



The resultant force acting on the body is zero

$$F_w - w = ma$$
$$a = 0$$
$$F_w - w = m \times 0$$
$$F_w - w = 0$$
$$F_w = w \longrightarrow (1)$$

This shows that the apparent weight of the object is equal to its real weight.

(b) When the elevator moving upward with uniform acceleration.

When the elevator moves upward with an acceleration a

$$\therefore \ F_w > w$$

The resultant force acting on the body is

$$F_w - w = ma$$
$$F_w = w + ma$$
$$F_w = mg + ma$$
$$F_w = m(g + a) \longrightarrow (2)$$

This shows that the apparent weight of the body has increased by an amount "ma"

(c) When the elevator moving downward with uniform acceleration.

When the elevator moves downward with an acceleration a

the resultant force is acting on the body is $w > F_w$

$$w - F_w = ma$$
$$F_w = w - ma$$
$$F_w = mg - ma$$
$$F_w = m(g - a) \longrightarrow (3)$$

This shows that the apparent weight of the body has decreased by an amount "m a".

(d) When the elevator falling freely under the action of gravity.

If the cable supporting the elevator breaks the elevator will fall down like a free body with an acceleration equal to g

$a = g$ putting is equation (iii)

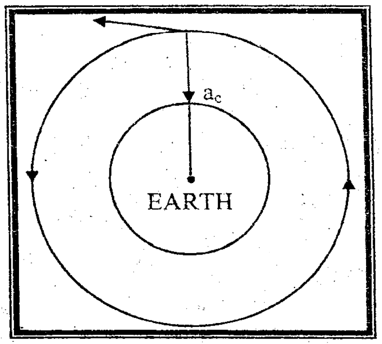
$$F_w = mg - ma$$
$$F_w = mg - mg$$
$$F_w = 0$$

Hence the spring balance will read the "zero" and the man in the elevator will think that the body has no weight i.e. it is in the state of "WEIGHTLESSENESS" besides the fact that the force of gravity still acts upon the body.

Conclusion:

In any freely falling frame of reference, apparent weight of all the bodies in it is "zero"

CAUSE OF WEIGHTLESSNESS IN SATELITE:



The concept of apparent weight can also be applied to the phenomenon of "weightlessness" in satellites bodies inside an orbiting satellite are not weightless; the earth's gravitational attraction continues to act on them just as though they were at rest relative to the earth but space ship in orbit has an acceleration $\vec{a}_\perp$ towards the centre of earth and equals to the value of acceleration due to gravity $\vec{g}$.

So, the apparent weight is given by

$$F_w = mg - ma_\perp$$

But $a_\perp = g$

$$F_w = mg - mg$$

$$F_w = 0$$

This shows that an astronaut or any other body in the space ship will be in the state of "weightlessness" or in state of "zero g". So a body released inside the space ship does not fall relative to it, and it appears to be weightless.

Importance of Artificial Gravity:

All the spacecrafts along with astronaut and other objects are in the state of free fall, and consequently will be in the state of weightlessness. Weightlessness in space crafts or satellites is highly in convenient to an astronavt in many ways. For example, we cannot pour liquid into a cup, neither we can drink from it. If the space craft is a space labortory intended to stay in space for a long time the weightlessness may be a severe handicap to the astronavt in performing experiments.

How to produce Artificial gravity?

In order to overcome these difficulties an artificial gravity is created in the space craft by spinning it around its own axis so that normal force of gravity can be supplied to the occupants in the space craft.

Derivation of relevant equation and calculation of frequency.

Calculation of spinning frequency:

To find the spinning frequency, consider a space craft consisting of two chambers connected by a tunnel of length "2R". If "υ" be the frequency of the tunnel and "T" be its time period then.

$$\upsilon = \frac{1}{T}$$

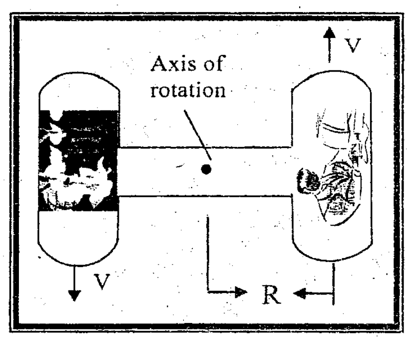
The magnitude of the centripetal acceleration "a_c" which is equal to the magnitude of the centrifugal acceleration is given by

$$a_c = \frac{4\pi^2 R}{T^2} \qquad \because r = R$$

OR $$a_c = 4\pi^2 R\upsilon^2$$

OR $$\upsilon^2 = \frac{a_c}{4\pi^2 R}$$

OR $$\upsilon = \frac{1}{2\pi}\sqrt{\frac{a_c}{R}} \qquad \text{-----------} \quad (1)$$



where R = half of the length of tunnel.

To create artificial gravity equal to the gravity at the surface of the earth, put $a_c = g$ in equ (1).

$$\boxed{\upsilon^2 = \frac{1}{2\pi}\sqrt{\frac{g}{R}}}$$

This is the relevant equation of spinning frequency

If length of the tunnel is 20m and g = 9.8 $\frac{m}{s^2}$ then

$$\upsilon = \frac{1}{2x3.142}\sqrt{\frac{9.8}{10}}$$

$$\upsilon = 0.158 \text{ rev / sec}$$

$$\upsilon = 0.158x\ 60 = 9.5 \text{ rev / min}$$

Thus the astronaut should feel comfortable in space craft if it is spinning at 9.5 revolutions/minute at a distance of 10m away from the axis of rotation as shown in figure.