TORQUE, ANGULAR MOMENTUM AND EQUILIBRIUM 5

TRANSLATORY MOTION:

If a body move in such a way that every particle of a body covers equal distance then motion of body is said to be "translatory motion". In translatory motion, it is not necessary that the body must move in straight line but its axes, fixed to the body, must remain parallel to itself at every instant. e.g.

1. Motion of a car on a straight road.

2. Motion of motor cyclist.

AXIS OF ROTATION:

The line about which a body rotates is called "axis of rotation".

ROTATORY MOTION:

If a body moves in such a way that every particle of a body moves in a circle about a line, called axis of rotation, then its motion is said to be *"ROTATORY MOTION"*. In rotatory motion every particle of rotating body does not cover equal distance.

KINDS OF ROTATORY MOTION:

There are two kinds of rotatory motion.

1. Spin Motion 2. Orbital Motion

(1) SPIN MOTION:

If the axis of rotation passes through the body of rotating object then its motion is said to be ***"SPIN MOTION"***. e.g.

1. Motion of a top

2. Motion of a wheel

3. Motion of an electric fan

4.Motion of the hands/needles of watch.

(2) ORBITAL MOTION:

If the axis of rotation does not pass through the body of rotating object then the motion is said to be "*ORBITAL MOTION*". e.g.

1. Motion of the planets around the sun

2. Motion of electrons around the nucleus

3. Circular motion of a stone tied to a string.

TORQUE:

The turning effect of force is called *"TORQUE"* or *"MOMENT OF FORCE"*.

OR

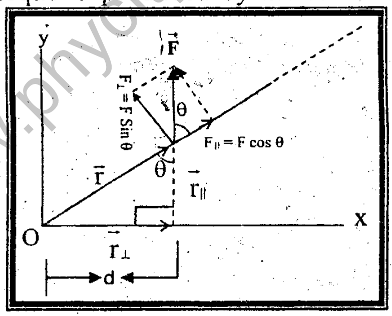
The physical quantity which produces angular acceleration is called "torque".

Consider a particle of mass "m" which is acted upon by a force $\vec{F}$. Let $\vec{r}$ be the position vector of application of the force, as shown in the figure. The force $\vec{F}$ can be resolved into components.

1. $F_{\parallel} = F\cos\theta$, parallel to $\vec{r}$.

2. $F_{\perp} = F\sin\theta$, perpendicular to $\vec{r}$.

The force $F_{\parallel}$ can pull the mass but can not rotate it, while the force $F_{\perp}$ can rotate the body about origin 'O'. Thus the Magnitude of torque $\vec{\tau}$ produced by the force $\vec{F}$ about origin O is given by,



$$\tau = r F_{\perp} = r F \sin\theta$$

Where θ = smaller angle between $\vec{r}$ & $\vec{F}$.

$$\vec{\tau} = r F \sin\theta \hat{n}$$

$\hat{n}$ = unit rector perpendicular to the plane containing $\vec{r}$ and $\vec{F}$

In vector from $\vec{\tau} = \vec{r} \times \vec{F}$.

Thus torque is the cross product of $\vec{r} \times \vec{F}$ The direction of torque is perpendicular to the plane containing $\vec{r}$ and $\vec{F}$ which can be found by right hand rule for cross product.

DETERMINANT FORM:

If x, y, z, be the co-ordinates of $\vec{r}$ & F_x, F_y, F_z, are the components of $\vec{F}$ along x, y & z axes respectively then,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

Thus torque can be written as.

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{J} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

The magnitude of torque can also be written as

$$\tau = Fr \sin\theta$$

$$\boxed{\tau = Fd}$$

Where $r\sin\theta$ = perpendicular distance between the line of action of force & axis of rotation and it is called "moment arm" "d".

Factors on which torque depends:

Torque depends upon two factors,

1. **The magnitude of the applied force.**
2. **The moment arm.**

The greater the force the greater will be the torque. Similarly the larger the moment arm greater will be the torque.

A body can rotate clockwise or anticlockwise. Hence a torque which produces counter clockwise rotation is considered as positive. The torque which produces clockwise rotation is considered as negative.

TORQUE DUE TO COUPLE:

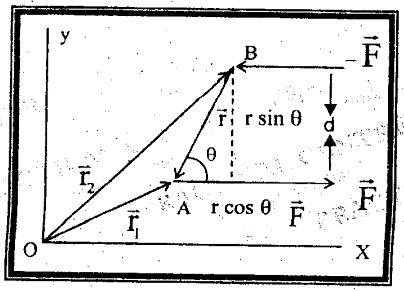
Two forces which are equal in magnitude but opposite in direction and not acting along the same line but acting on the same body constitutes a couple.

Consider a couple composed of two forces $\vec{F}$ & $-\vec{F}$ acting at point A & B of a body, as shown in figure.

If $\vec{\tau}_1$ is the torque due to $\vec{F}$ then

$\vec{\tau}_1 = \vec{r}_1 \times \vec{F}$ Similarly, if $\vec{\tau}_2$ is the torque due to $-\vec{F}$ then

$$\vec{\tau}_2 = \vec{r}_2 \times (-\vec{F})$$



The total torque $\vec{\tau}$ of the couple will be

$$\vec{\tau} = \vec{\tau} + \vec{\tau}_2$$

$$\vec{\tau} = \vec{r}_1 \times \vec{F} + \vec{r}_2 \times (-\vec{F})$$

$$= \vec{r}_1 \times \vec{F} - \vec{r}_2 \times \vec{F}$$

$$\vec{\tau} = (\vec{r}_1 - \vec{r}_2) \times \vec{F} \longrightarrow (1)$$

where $\vec{r}_1$ & $\vec{r}_2$ are the position vectors of points A & B respectively, as shown in the figure.

If $\vec{r}$ be the distance between A & B then by head to tail rule

$$\vec{r}_2 + \vec{r} = \vec{r}_1$$

OR $$\vec{r} = \vec{r}_1 - \vec{r}_2$$

Thus eq (1) become $$\vec{\tau} = \vec{r} \times \vec{F}$$

The magnitude of Torque $= \tau = rF\sin(180-\theta) = Fr\sin\theta \quad \therefore \sin(180-\theta) = \sin\theta$

If we put $r \sin\theta = d$, then

$$\tau = F\,d \longrightarrow (2)$$

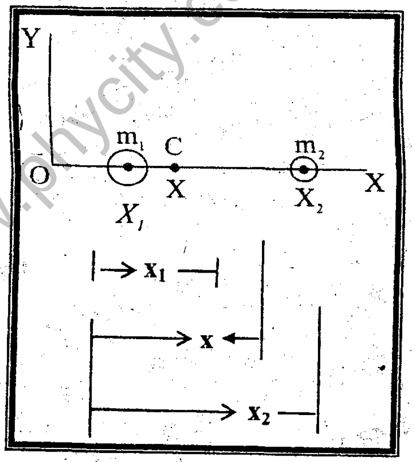
The perpendicular distance $d = r \sin\theta$ between the forces forming couple is called "MOMENT ARM OF THE COUPLE". From eq(2) it is concluded that Magnitude of Torque = Magnitude of any of the forces x moment arm of the couple.

The direction of torque is perpendicular to the plane containing $\vec{r}$ & $\vec{F}$ & can be found by right hand rule for cross product.

CENTRE OF MASS:

Definition:

The whole mass of a body can be considered to act at a single point know as centre of mass of body. If a body is symmetrical and of uniform composition the centre of mass is at the geometrical centre of the body.



If the two masses are placed at distances x_1 and x_2 and point C is at distance 'x' from O such that the two masses together produce same torque as produced by individuals, then this point C is known as the "centre of mass". The torque "τ" produced by the combined mass $(m_1 + m_2)$ is given by the $\tau = (m_1 + m_2)g\,x$ ------(1)

The torque of weight of m_1 is $\tau_1 = w_1\,x_1$ and the torque of weight of m_2 is $\tau_2 = w_2\,x_2$ then net torque of these particles will be

$$\tau = \bar{\tau}_1 + \bar{\tau}_2$$
$$\tau = w_1 x_1 + w_2 x_2$$
$$\tau = (m_1 x_1 + m_2 x_2)g \text{ \_\_\_\_\_\_(2)}$$

Comparing eq (1) & (2)

$$(m_1 + m_2)\,g\;x = (m_1\,x_1 + m_2\,x_2)\,g$$

$$x = \frac{(m_1\,x_1 + m_2\,x_2)\,g}{(m_1 + m_2)\,g}$$

$$\therefore \boxed{x = \frac{m_1\,x_1 + m_2\,x_2}{m_1 + m_2}}$$

In three dimentional space if there are more than two masses at various locations & xc, yc & zc are the coordinates of the centre of mass then these are given by

$$xc = \frac{m_1 x_1 + m_2 x_2 + -----}{m_1 + m_2 + -----} = \frac{\Sigma m_i x_i}{\Sigma m_i}$$

$$yc = \frac{m_1 y_1 + m_2 y_2 + -----}{m_1 + m_2 + -----} = \frac{\Sigma m_i y_i}{\Sigma m_i}$$

$$zc = \frac{m_1 z_1 + m_2 z_2 + ------}{m_1 + m_2 -----} = \frac{\Sigma m_i z_i}{\Sigma m_i}$$

Where x_i, y_i & z_i are the coordinates of the particle of mass m_i.

If the object is completely in uniform gravitational field, the centre of gravity of an extended body coincides with its centre of mass.

If this is not the case, the centre of gravity does not coincide with the centre of mass.

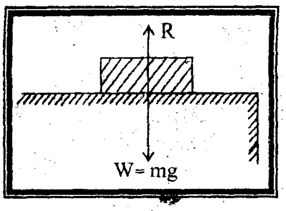
EQUILIBRIUM:

A body is said to be in state of equilibrium if it is at rest or moving with a uniform velocity & the body does not posses any acceleration, neither linear nor angular.

There are two types of equilibrium.

(i) Static Equilibrium

(ii) Dynamic Equilibrium

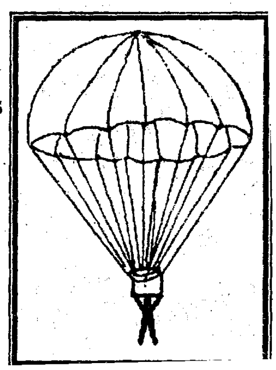


(i) STATIC EQUILIBRIUM:

A body at rest is said to be in state of "STATIC EQUILIBRIUM".

Example:

1. A book lying on the table.
2. A pole of street light.
3. Any thing in rest with any support is example of Static Equilibrium.



(ii) DYNAMIC EQUILIBRIUM:

If body is in uniform motion along a straight line then it is said to be in "DYNAMIC EQUILIBRIUM".

Example:

1. Jumping of a paratrooper.
2. A moving steel ball in a tank of viscous liquid.
3. Planet in the space.

FIRST CONDITION OF EQUILIBRIUM:

First condition of equilibrium states that "a body will be in equilibrium if the resultant of (vector sum of) all the forces acting on it is zero".

Explanation:

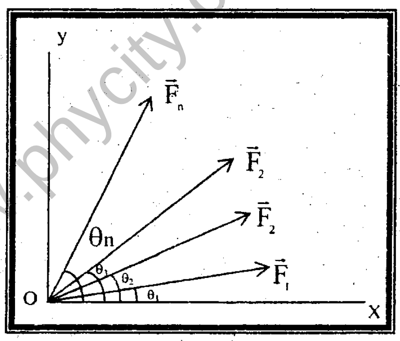
Consider a body on which "n" external forces $\vec{F}_1, \vec{F}_2 ----- \vec{F}_n$ are acting. By the first condition of equilibrium, we can say

$$\vec{F}_1 + \vec{F}_2 + ---- + \vec{F}_n = 0 \longrightarrow (1)$$

OR

$$\Sigma\ \vec{F} = 0$$

If these forces are in plane, say the x-y plane, then we can resolve the above forces in x-y components



Thus

$$\vec{F}_1 = \vec{F}_{1x} + \vec{F}_{1y} = F_{1x}\hat{i} + F_{1y}\hat{j}$$

$$\vec{F}_2 = \vec{F}_{2x} + \vec{F}_{2y} = F_{2x}\hat{i} + F_{2y}\hat{j}$$

$$\vec{F}_n = \vec{F}_{nx} + \vec{F}_{ny} = F_{nx}\hat{i} + F_{ny}\hat{j}$$

Putting the above values of

$\vec{F}_1, \vec{F}_2$, ------ $\vec{F}_n$ in eq (1)

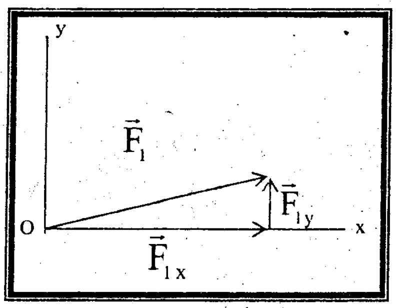
$$(F_{1x}\hat{i} + F_{1y}\hat{j}) + (F_{2x}\hat{i} + F_{2y}\hat{j}) + ---- + (F_{nx}\hat{i} + F_{ny}\hat{j}) = 0$$

$$(F_{1x} + F_{2x} + ----- + F_{nx})\hat{i} + (F_{1y} + F_{2y} + ----- + F_{ny})\hat{j} = 0\hat{i} + 0\hat{j}$$

By equating the Components,

$$\therefore\ F_{1x} + F_{2x} + ---- + F_{nx} = 0 \longrightarrow (2)$$

OR $\sum_{i=1}^{n} F_{ix} = 0$



Also $F_{1y} + F_{2y} + ------ + F_{ny} = 0 \longrightarrow (3)$

OR $\sum_{i=1}^{n} F_{iy} = 0$

SECOND CONDITION OF EQUILIBRIUM:

The second condition of equilibrium states that "a body is said to be in state of rotational equilibrium if the vector sum of all the torques acting on it is zero".

EXPLANATION:

Consider a body on which 'n' torques $\vec{\tau}_1, \vec{\tau}_2 --- \vec{\tau}_n$ are acting about any axis. By the 2nd condition of equilibrium, we can write

$$\vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + ------ \vec{\tau}_n = 0$$

OR

$$\sum_{i=1}^{n} \tau_i = 0$$

Simply $\Sigma \tau = 0$

CONCLUSION:

For a body to be in complete equilibrium there should be no linear acceleration & angular acceleration i.e. conditions for equilibrium are

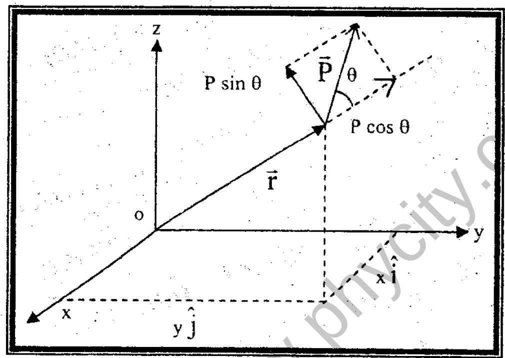
$\sum F_X = 0$

$\sum F_y = 0$

$\sum \tau = 0$

ANGULAR MOMENTUM:

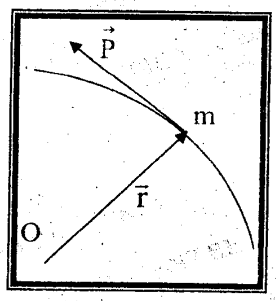
The angular momentum of a particle about a fixed point is defined as "It is a vector product of. Position vector of the particle with respect to a point and its linear momentum.



Consider a particle of mass "m" whose position is given by position vector "$\vec{r}$" with respect to origin 'O'

Let "$\vec{p}$" be the liner momentum of the particle and it is making an angle "θ" with " " as shown in the figure.

If $\vec{P}$ is resolved into two components "P cos θ" and "P sin θ" Then "P cos θ" can move the mass "m" along "$\vec{r}$" but can not rotate it about "O" while "P sin θ" can rotate the mass "m" about "O" hence angular momentum will be produced. The magnitude of angular momentum is given by $\vec{L}$



$L = r P \sin\theta \longrightarrow (1)$

$\vec{L} = \vec{r} \times \vec{P} \longrightarrow (2)$

Thus angular momentum is the cross product of "$\vec{r}$" & "$\vec{F}$".

The magnitude of linear momentum $\vec{p}$ is given by $P = m v$.

Thus eq (1) becomes

$L = r m v \sin\theta$

$L = m r v \sin\theta$

$\vec{L} = m(\vec{r} \times \vec{p})$

The direction of angular momentum is perpendicular to both $\vec{r}$ & $\vec{P}$ and it can be found by right hand rule for cross product.

For circular motion $\vec{r}$ & $\vec{P}$ are perpendicular to each other.

$\therefore \theta = 90^o$ & $\sin 90^o = 1$

Thus eq (1) becomes

$L = r p$

OR $L = m r v$

IN THE FORM OF COMPONENTS:

We have $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ **and** $\vec{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$

Thus equation (2) can be written as

$$\vec{L} = \vec{r} \times \vec{P} = (x\hat{i} + y\hat{j} + z\hat{k}) \times (p_x\hat{i} + p_y\hat{j} + p_z\hat{k})$$

OR

$$\vec{L} = \vec{r} \times \vec{P} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \hat{i}\begin{vmatrix} y & z \\ p_y & p_z \end{vmatrix} + \hat{j}\begin{vmatrix} z & x \\ p_z & p_x \end{vmatrix} + \hat{k}\begin{vmatrix} x & y \\ p_x & p_y \end{vmatrix}$$

$$L_x\hat{i} + L_y\hat{j} + L_z\hat{k} = \hat{i}(yp_z - zp_y) + \hat{j}(zp_x - xp_z) + \hat{k}(xp_y - yp_x)$$

The scalar components of angular momentum are

$$L_x = yP_z - zP_Y$$
$$L_y = zP_x - xP_z$$
$$L_z = xP_y - yP_x$$

DIMENSIONS:

Dimensions of angular momentum are

$$\Rightarrow \frac{ML^2}{T^2}$$
$$\Rightarrow ML^2T^{-2}$$

UNIT:

In system international (S.I) the unit of mass is Kg, unit of displacement is metre and velocity is m/s. Therefore unit of angular momentum will be

$$= Kg\frac{m}{s} \times \frac{s}{s} \qquad \because\ Kg\frac{m}{s^2} = \text{Newton}$$
$$= \left(Kg\frac{m}{s^2}\right) \times ms \qquad \because\ Nm = \text{Joule}$$
$$= (Nm)s$$
$$= Js$$

Thus unit of angular momentum is Joule-second (JS).

LAW OF CONSERVATION OF ANGULAR MOMENTUM:

According to this law "The angular momentum of a particle or a system of particles is conserved (constant) if the net external torque acting on it is zero".

PROOF:

According to Newton's second law of motion, the net force acting on a particle of mass ' m ' moving with an instantaneous velocity $\vec{v}$ is equal to time rate of change of its linear momentum.

$$\vec{F} = \frac{d}{dt}(\vec{P})$$

Taking cross product with $\vec{r}$ both sides we get

$$\vec{r}\times\vec{F}=\vec{r}\times\frac{d}{dt}(\vec{P}) \qquad \because \vec{\tau}=\vec{r}\times\vec{F}$$

But by definition $\vec{r}\times\vec{F}$ is the torque acting on the particle.

$$\vec{\tau}=\vec{r}\times\frac{d}{dt}(\vec{P}) \rightarrow (i)$$

We know that the angular momentum $\vec{\ell}$ is given as

$$\vec{\ell}=\vec{r}\times\vec{P}$$

Differentiating both sides with respect to time.

$$\frac{d\vec{\ell}}{dt}=\vec{r}\times\frac{d\vec{P}}{dt}+\frac{d\vec{r}}{dt}\times\vec{P}$$

From equation (i)

$$\frac{d\vec{\ell}}{dt}=\vec{\tau}+m(\vec{v}\times\vec{v}) \qquad \because \vec{P}=m\vec{v}$$

$$\frac{d\vec{r}}{dt}=\vec{v}$$

$$\frac{d\vec{\ell}}{dt}=\vec{\tau}+m\times 0 \qquad \because \vec{v}\times\vec{v}=0$$

$$\frac{d\vec{\ell}}{dt}=\vec{\tau}\rightarrow (ii)$$

If the net external torque acting on the particle is zero. Then eq. (ii) becomes,

$$\frac{d\vec{\ell}}{dt}=0$$

Integrating, we get.

$$\boxed{\vec{\ell}=\text{Constant}}$$

FOR SYSTEM OF PARTICLES:

Thus the angular momentum of a particle is conserved (constant) if the net torque acting on it is zero.

Consider a system of '**n**' particles which is acted upon by a number of external forces. The total angular momentum of the system is given as.

$$\vec{L}=\vec{\ell}_1+\vec{\ell}_2+\vec{\ell}_3+ ------ +\ell_n$$

$$\vec{L}=\vec{r}_1\times\vec{P}+\vec{r}_2\times\vec{P}_2+ ------ +\vec{r}_n\times\vec{P}_n$$

Differentiating both sides with respect to time

$$\frac{d\vec{L}}{dt}=\frac{d}{dt}\left(\vec{r}_1\times\vec{P}_1\right)+\frac{d}{dt}\left(\vec{r}_2\times\vec{P}_2\right)+ ------ +\frac{d}{dt}\left(\vec{r}_n\times\vec{P}_n\right)$$

$$\frac{dL}{dt}=\vec{r}_1\times\frac{d\vec{P}_1}{dt}+\frac{d\vec{r}_1}{dt}\times\vec{P}_1+\vec{r}_2\times\frac{d\vec{P}_2}{dt}+\frac{d\vec{r}_2}{dt}\times\vec{P}_2+ ------$$

$$------+\vec{r}_n\times\frac{d\vec{P}_n}{dt}+\frac{d\vec{r}_n}{dt}\times\vec{P}_n$$

$\vec{r}_1 x \frac{d\vec{P}_1}{dt} = \vec{\tau}_1 \qquad \vec{r}_2 x \frac{d\vec{P}_2}{dt} = \vec{\tau}_2$

$\frac{d\vec{L}}{dt} = \vec{\tau}_1 + m_1(\vec{v}_1 x \vec{v}_1) + \vec{\tau}_2 + m_2(\vec{v}_2 x \vec{v}_2) + ------ + \vec{\tau}_n + m_n(\vec{v}_n x \vec{v}_n)$

$\vec{v}_1 x \vec{v}_1 = 0 \qquad \vec{v}_2 x \vec{v}_2 = 0 \quad \vec{v}_n x \vec{v}_n = 0$

$\frac{d\vec{L}}{dt} = \vec{\tau}_1 + m_1 x0 + \vec{\tau}_2 + m_2 x0 + ------ + \vec{\tau}_n + m_n x0$

$\frac{d\vec{L}}{dt} = \vec{\tau}_1 + \vec{\tau}_1 + ------ + \vec{\tau}_n$

$\vec{\tau}_1 + \vec{\tau}_2 + ------ + \vec{\tau}_n = \vec{\tau}$

$\frac{d\vec{L}}{dt} = \vec{\tau}$

If the net external torque acting on the system is zero. Then

$\frac{d\vec{L}}{dt} = 0$

Integrating, we get.

$\boxed{\vec{L} = \text{Constant}}$

LOCATION OF AXIS:

"If the body is in translational equilibrium, the net torque is zero at any other point."

In order to prove the above statement, consider a uniform bar of length "L" resting under two pegs at its ends, as shown in fig (1).

Three forces are acting on the bar.

1. $\vec{F}_1$ = Force exerted by one peg in upward direction.
2. $\vec{F}_2$ = Force exerted by the other peg in upward direction.
3. $\vec{F}_3$ = Force of gravity acting downward.

Since the bar is in equilibrium.

$\therefore \Sigma F_x = 0$

$\Sigma F_y = 0$

$F_1 + F_2 - w = 0 \quad \text{———} \quad (1)$

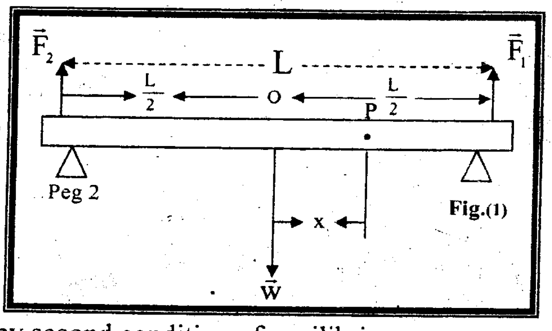


Fig.(1)

If "O" be the axis of rotation, then by second condition of equilibrium.

$\Sigma\tau = 0$

$F_1 x \frac{L}{2} - F_2 x \frac{L}{2} = 0$

$\frac{L}{2}(F_1 - F_2) = 0$

$F_1 - F_2 = 0 \longrightarrow (2)$

If we take 'p' instead of 'O' at distance "x" from "O", as axis of rotation then.

$$\Sigma\tau = F_1(\frac{L}{2} - x) + wx - F_2(\frac{L}{2} + x)$$
$$= F_1\frac{L}{2} - F_1x + wx - F_2\frac{L}{2} - F_2x$$
$$= \frac{L}{2}(F_1 - F_2) - x\,(F_1 + F_2 - w)$$

But from eq (2)

$$F_1 - F_2 = 0$$

& From eq (1)

$$F_1 + F_2 - w = 0$$

$$\therefore \Sigma\tau = \frac{L}{2} \times 0 - x\,(0) = 0$$
$$\Sigma\tau = 0$$

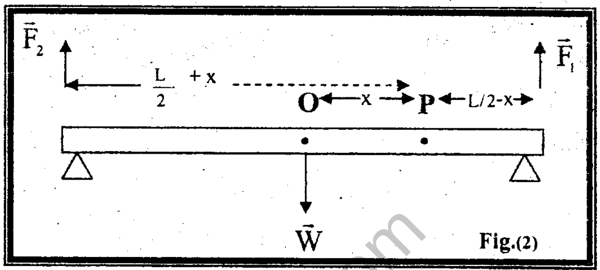


Fig.(2)

This shows that the torque acting on a body is independent of origin.

Q. **Prove that torque** $\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$

Where

And $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

The torque $\vec{\tau}$ produce by the force $\vec{F}$ about origin O is given $\vec{\tau} = \vec{r} \times \vec{F}$

$$\vec{\tau} = (x\hat{i} + y\hat{j} + z\hat{k}) \times (Fx\hat{i} + Fy\hat{j} + Fz\hat{k})$$
$$\vec{\tau} = xFx(\hat{i}\times\hat{i}) + xFy\,(\hat{i}\times\hat{j}) + xFz(\hat{i}\times\hat{k}) + yFx\,(\hat{j}\times\hat{i}) + yFy\,(\hat{j}\times\hat{j}) + yFz\,(\hat{j}\times\hat{k})$$
$$+ zFx(\hat{k}\times\hat{i}) + zFy\,(\hat{k}\times\hat{j}) + zFz\,(\hat{k}\times\hat{k})$$
$$\because \hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$$
$$\hat{i}\times\hat{j} = \hat{k} \quad \hat{i}\times\hat{k} = -\hat{j} \quad \hat{j}\times\hat{i} = -\hat{k} \quad \hat{j}\times\hat{k} = \hat{i}$$
$$\hat{k}\times\hat{i} = \hat{j} \quad \hat{k}\times\hat{j} = \hat{i}$$

$$\vec{\tau} = xFx(0) + xFy\,(\hat{k}) + xFz(-\hat{j}) + yFx\,(-\hat{k}) + yFy\,(0) + yFz\,(\hat{i}) + zFx\,(\hat{j}) + zFy\,(-\hat{i}) + yFz\,(0)$$
$$\vec{\tau} = xFy\,\hat{k} - xFz\,\hat{j} - yFx\,\hat{k} + yFz\,\hat{i} + zFx\,\hat{j} - zFy\,\hat{i}$$
$$\vec{\tau} = yFz\,\hat{i} - zFy\,\hat{i} - xFz\,\hat{j} + zFx\,\hat{j} + xFy\,\hat{k} - yFx\,\hat{k}$$
$$\vec{\tau} = \hat{i}\,(yFz - zFy) - \hat{j}\,(xFz - zFx) + \hat{k}\,(xFy - yFx)$$
$$\vec{\tau} = \hat{i}\begin{vmatrix} y & z \\ Fy & Fz \end{vmatrix} - \hat{j}\begin{vmatrix} x & z \\ Fx & Fz \end{vmatrix} + \hat{k}\begin{vmatrix} x & y \\ Fx & Fy \end{vmatrix}$$
$$\vec{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & k \\ x & y & z \\ Fx & Fy & Fz \end{vmatrix}$$ **Proved.**