MOTION IN TWO DIMENSIONS 4

Projectile:

A body thrown at some angle θ above or below the horizontal such that ($0 < \theta < \pi/2$) and moves freely under the action of gravity is called projectile.

Projectile Motion:

The motion in which a body has constant horizontal component of velocity but changing vertical component of velocity is called projectile motion.

Examples:

1. Motion of a football kicked by a player.
2. Motion of a bomb released from a horizontally flying airplane.

Assumptions for Projectile Motion:

1- The acceleration due to gravity "g" is constant over the range of motion and is directed downward.

2- The effect of air resistance is negligible.

3- Rotation of the earth does not affect the motion.

Consider a projectile shot with velocity $\vec{V}_0$ at an angle θ with the horizontal, as shown in the figure. The initial velocity $\vec{V}_0$ of the projectile can be resolved into two rectangular components.

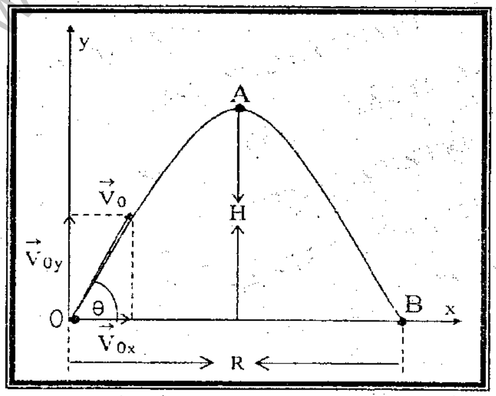
1- $\vec{V}_{0x}$ along horizontal axis

2- $\vec{V}_{0y}$ along vertical axis

The magnitude of these components are given by:

$$V_{0x} = V_0 \cos\theta$$

$$V_{0y} = V_0 \sin\theta$$



If air resistance is negligible and the rotation of earth does not affect the motion, then horizontal velocity at any instant "t" will remain constant and is given by,

$$V_x = V_{0x} = V_0 \cos\theta$$

The vertical velocity will decrease during upward motion and increase during downward motion. Final vertical velocity "V_y" at any instant "t" is given by.

$$V_{fy} = V_{iy} + a_y t$$

$$V_{fy} = V_y$$

$$V_{iy} = V_{oy} = V_o \sin\theta$$

$$a_y = -g$$

$$V_y = V_{oy} - gt$$

OR $\boxed{V_y = V_o \sin\theta - gt}$ (Upward motion)

OR $\boxed{V_y = V_o \sin\theta + gt}$ (Downward motion)

TIME TAKEN BY THE PROJECTILE TO REACH THE MAXIMUM HEIGHT:

Definition:

It is the time required by a projectile to reach its maximum height.

Derivation:

If "t" be the time taken by the projectile to reach the maximum height then it can be found by using formula:

$$V_{fy} = V_{iy} + a_y t$$

Where $V_{iy} = V_{oy} = V_0 \text{ Sin } \theta$

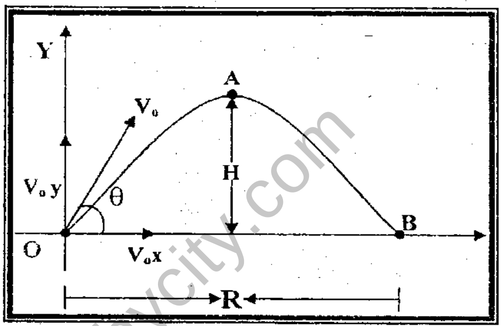
$$a_y = -g$$

$$V_{fy} = V_y = 0$$

$$0 = V_0 \text{ Sin } \theta - gt$$

$$gt = V_0 \text{ Sin } \theta$$

$$\boxed{t = \frac{V_0 \text{ Sin } \theta}{g}} \text{ ------ (1)}$$



MAXIMUM HEIGHT REACHED BY THE PROJECTILE:

Definition:

The maximum vertical distance attained by the projectile is called maximum height. It is denoted by "H".

Derivation:

The maximum height reached by the projectile can be found by considering vertical motion from O to A.

$$Y = V_{iy}t + \frac{1}{2} a_y t^2$$

Where $V_{iy} = V_{0y} = V_0 \text{ Sin } \theta$

$$t = \frac{V_0 \text{Sin}\theta}{g}$$

$$a_y = -g$$

$$Y = H$$

$$H = V_0 \text{ Sin } \theta \frac{V_0 \text{Sin}\theta}{g} + \frac{1}{2}(-g)\left(\frac{V_0 \text{Sin}\theta}{g}\right)^2$$

$$H = \frac{V_0^2 \text{Sin}^2\theta}{g} - \frac{1}{2} g \frac{V_0^2 \text{Sin}^2\theta}{g^2}$$

$$H = \frac{V_0^2 \text{ Sin}^2\theta}{g} - \frac{V_0^2 \text{ Sin}^2\theta}{2g}$$

$$H = \frac{2V_0^2 \text{ Sin}^2\theta - V_0^2 \text{ Sin}^2\theta}{2g}$$

$$\boxed{H = \frac{V_0^2 \text{ Sin}^2\theta}{2g}} \longrightarrow (2)$$

TOTAL TIME OF FLIGHT OF PROJECTILE:

Definition:

The time during which projectile remains in air is called its Total time of flight 'T'.

Derivation:

It can be found by formula by considering its vertical motion from O to B.

$$Y = V_{iy}\,t + \frac{1}{2}a_y t^2$$

Where

$$Y = 0$$
$$V_{iy} = V_{0y} = V_0 \sin\theta$$
$$a_y = -g$$
$$t = T$$

$$0 = V_0 \sin\theta\, T + \frac{1}{2}(-g)T^2$$

$$0 = V_0 \sin\theta\, T - \frac{1}{2}gT^2$$

$$\frac{1}{2}gT^2 = V_0 \sin\theta\, T$$

$$\frac{1}{2}gT = V_0 \sin\theta$$

$$\boxed{T = \frac{2V_0 \sin\theta}{g}} \longrightarrow (3)$$

RANGE OF THE PROJECTILE:

Definition:

The horizontal distance covered by the projectile between point of projection and point of return to level of projection is called 'Range' of the projectile and is represented by "R".

Derivation:

It can be found by formula by considering its horizontal motion form O to B.

$$X = V_{ix}\,t + \frac{1}{2}a_x t^2$$

$$X = R$$
$$V_{ix} = V_0 \cos\theta$$
$$t = T = \frac{2V_0 \sin\theta}{g}$$
$$a_x = 0$$

$$R = V_0 \cos\theta \times \frac{2V_0 \sin\theta}{g} + \frac{1}{2} \times 0 \times \left[\frac{2V_0 \sin\theta}{g}\right]^2$$

$$R = \frac{V_0^2(2\sin\theta\cos\theta)}{g} + 0$$

We know that $\sin 2\theta = 2\sin\theta\cos\theta$

$$\boxed{R = \frac{V_0^2}{g}\sin 2\theta} \text{ --------------------(4)}$$

The Maximum Range:

As $R = \dfrac{V_0^2 \sin 2\theta}{g}$ $\longrightarrow$ (5)

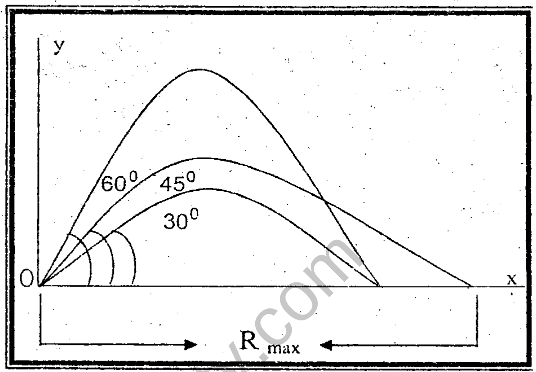
If "V_0" and "g" are constants, then equation (5) shows that "R" is maximum when the value of sin2θ is maximum.

The maximum value of $\sin 2\theta = 1$

$$2\theta = \sin^{-1}(1)$$

$$2\theta = 90°$$

$$\theta = 45°$$



putting $R = R_{max}$ and $\theta = 45^0$ in eq (5)

$$R_{max} = \frac{V_0^2 \sin 2 \times 45^0}{g}$$

$$R_{max} = \frac{V_0^2 \sin 90^0}{g} \qquad \because \sin 90° = 1$$

$$\boxed{R_{max} = \frac{V_0^2}{g}} \longrightarrow (4)$$

Thus for maximum horizontal range the angle of projection should be 45°.

Projectile Trajectory OR Position of Projectile:

The path followed by a projectile is referred as its ***"trajectory"***. The position of a projectile at any instant "t" can be found by knowing its horizontal displacement "X" and vertical displacement "Y".

The horizontal displacement "X" is given by:

$$X = V_{0x}t$$

$$X = V_0 \cos\theta \, t$$

OR $t = \dfrac{X}{V_0 \cos\theta}$ ----------------------(i)

The vertical displacement "y" is given by:

$$Y = V_{oy}t - \frac{1}{2}gt^2$$

OR $Y = (V_0 \sin\theta)t - \frac{1}{2}gt^2$

By putting the value of "t" from equation (i), we get

$$Y = V_0 \sin\theta \left(\frac{X}{V_0 \cos\theta}\right) - \frac{g}{2}\left(\frac{X}{V_0 \cos\theta}\right)^2$$

$$Y = X \frac{\sin\theta}{\cos\theta} - \frac{X^2}{2} \times \frac{g}{V_0^2 \cos^2\theta}$$

$$Y = X \tan\theta - \frac{1}{2}\left(\frac{g}{V_0^2}\sec^2\theta\right)X^2$$

For a given value of "θ" & the initial velocity of the projectile "V_0", "tanθ", "Secθ" & "g" are constants.

Therefore if we put

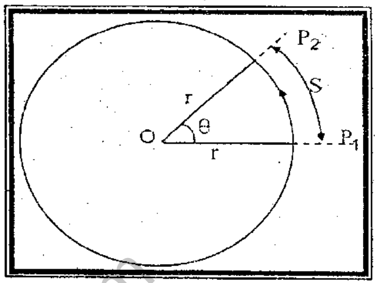
$a = \tan\theta$

& $\quad b = \dfrac{g}{V_0^2}\text{Sec}^2\theta$ we get

$$\boxed{Y = ax - \frac{1}{2}bx^2}$$

This equation shows that trajectory of a projectile is a parabola as if the standard equation of parabola.

Angular Displacement:



Consider an object moving along circular path of radius "r" consider further that the object initially at the point P_1 on the circumference of the circle. After a small interval of time, it moves to the position P_2. The angle P_1OP_2 or θ is called ***Angular Displacement*** of the particle or the object.

Measurement of Angular Displacement:

The angular displacement is measured in degrees. Another unit for the measurement of angular displacement is ***Radian.***

Degree:

If the circumference of a circle is divided into 360 equal parts, the angle made by each part at the centre of the circle is called one degree.

Radian:

It is the angle made at the centre of a circle where the arc length 's' is equal to the radius of the circle.

Arc P_1P_2 = r = radius of circle

$\angle P_1O\,P_2$ = 1 Radian

Angular displacement in radian can be found by using formula.

$$\text{Angular displacement} = \frac{\text{Length of arc}}{\text{Radius of the circle}}$$

If length of the arc $P_1\,P_2$ = S, radius of the circle = r, then angular displacement θ in radian is given by:

$$\theta = \frac{S}{r}$$

OR $\boxed{S = r\,\theta}$

Relation Between Degree & Radian:

When a particle completes one revolution, its angular displacement is equal to 360° i.e.

$$\theta = 360° \text{ ---------------------- } \textbf{(1)}$$

In one revolution length of the arc "S" is equal to the circumference of the circle.

i.e. $\quad S = 2\pi r$

$$\therefore \quad \theta \text{ (in radian)} = \frac{2\pi \cancel{r}}{\cancel{r}}$$

$$\theta = 2\pi \text{ radian ---------------------- } \textbf{(2)}$$

comparing equation (1) & (2)

2π radian = 360°

$$1 \text{ radian} = \frac{360}{2\pi} = \frac{360^0}{2 \times \frac{22}{7}} = \frac{7 \times 360}{44} = 57.3^0$$

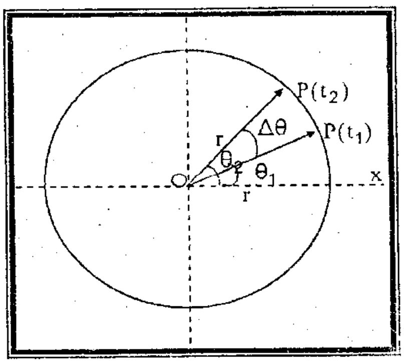
$$\text{similarly, } 1^0 = \frac{2\pi}{360} = \frac{2 \times 22}{360 \times 7} = 0.01745 \text{ radian}$$

Angular Velocity:

The angular displacement covered by a body in unit time or rate of change of angular displacement is called "***Angular Velocity***". It is denoted by "$\vec{\omega}$".

Consider a body moving counter clockwise in a circle of radius r. If the angular position of P is at a time "t_1" is θ_1 & at a later time "t_2" its angular position is θ_2 with is respect to the x-axis then the change in angular displacement

$$\theta_2 - \theta_1 = \Delta\theta$$



The time interval $t_2 - t_1 = \Delta t$. The magnitude of average angular velocity i.e. average angular speed "ω_{av}" is given by: $\omega_{av} = \dfrac{\Delta\theta}{\Delta t}$

Instantaneous Angular Velocity:

The angular velocity of a body at any instant is called instantaneous angular velocity.

$$\omega_{ins} = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}$$

Unit:

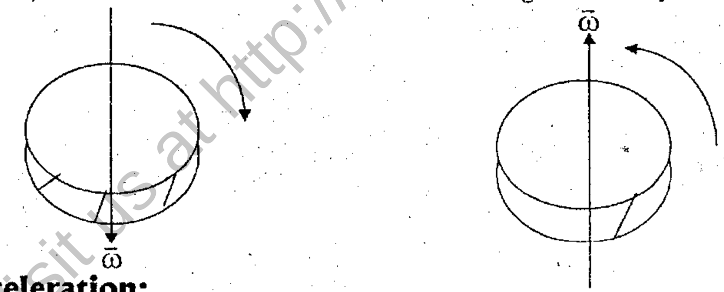
(1) Radian per second [S·I Unit]
(2) Degree per Second [B·E System]
(3) Revolution per Second

Direction:

The direction of angular velocity can be found by right hand rule.

Right Hand Rule:

If we hold the axis of rotation in right hand and rotate the fingers of right hand along the direction of rotation, then the thumb will show the direction of angular velocity.



Angular Acceleration:

The rate of change of angular velocity is called ***"angular acceleration"***. It is denoted by "$\vec{\alpha}$".

Consider a body moving in a circle of radius "r". If ω_1 & ω_2 be the magnitudes of the instantaneous angular velocities of the body at t_1 & t_2 respectively then average angular acceleration "α_{av}" is given by:

$$\alpha_{av} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

Instantaneous Angular Acceleration:

The angular acceleration of a body at any instant, is called instantaneous angular acceleration.

$$\alpha_{ins} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$$

Unit:

(1) Radian per second (rad/s^2) [S·I Unit]
(2) Degree per second per second (deg/s^2) [B·E System]
(3) Revolution per second per second (rev./s^2)

Direction:

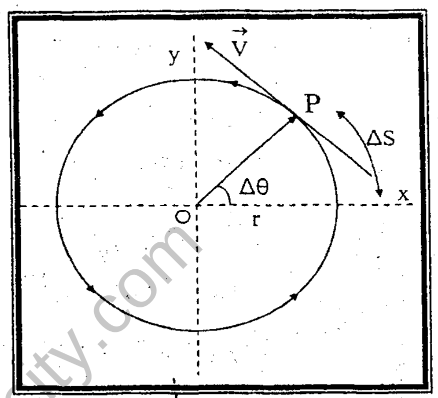
If ω is increasing then α is in the direction of ω and if ω is decreasing then α is opposite to the direction of ω

Relation Between Angular and Linear Quantities:

1-Relation for Velocity:

Consider a particle "P" in an object (wheel) rotating along a circular path of radius "r" about an axis O, (axle of the wheel), perpendicular to the plane of the paper, as shown in the figure.

Suppose the particle "P" rotates through an angle "$\Delta\theta$" in a time "Δt". If "ΔS" is the length of the arc described by the particle then:



$$\Delta\theta = \frac{\Delta S}{r}$$

Dividing both sides by Δt,

$$\frac{\Delta\theta}{\Delta t} = \frac{1}{r}\frac{\Delta S}{\Delta t}$$

OR $$\frac{\Delta S}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$$

If Δt approaches to zero then we can write

$$\lim_{\Delta t \to 0}\frac{\Delta S}{\Delta t} = r\lim_{\Delta t \to 0}\frac{\Delta\theta}{\Delta t}$$

But $\lim_{\Delta t \to 0}\frac{\Delta S}{\Delta t} = V_{ins}$ = instantaneous linear speed.

And $\lim_{\Delta t \to 0}\frac{\Delta\theta}{\Delta t} = \omega_{ins}$ = instantaneous angular speed.

$V_{ins} = r\omega_{ins}$ If the particle is with constant angular velocity "ω" then;

$$V = r\,\omega$$

Tangential Velocity: The velocity, which is tangent to the circular path, is called ***tangential velocity.*** It is given by: $V_t = r\,\omega$

2-Relation for Tangential Acceleration:

The acceleration of a particle which is tangent to the circular path is called ***"tangential acceleration".*** Suppose a particle rotating about a fixed axis, changes its angular velocity by "$\Delta\omega$" in a time "Δt". Then the change in its tangential velocity, (linear velocity), "ΔV_t" at the end of this interval is:

$$\Delta V_t = r\Delta\,\omega$$

Dividing both sides by Δt

$$\frac{\Delta V_t}{\Delta t} = r\frac{\Delta\omega}{\Delta t}$$

If "Δt" approaches to zero then:

$$\lim_{\Delta t \to 0}\frac{\Delta V_t}{\Delta t} = r\lim_{\Delta t \to 0}\frac{\Delta\omega}{\Delta t}$$

But $\lim_{\Delta t \to 0}\frac{\Delta V_t}{\Delta t} = a_{ins}$ = instantaneous tangential

Acceleration or instantaneous linear acceleration "a_{ins}" &

$\lim_{\Delta t \to 0}\frac{\Delta V_t}{\Delta t} = a_{ins}$ = instantaneous angular acceleration α then:

$$\boxed{a = a_t = r\,\alpha}$$

In vector form $\vec{a} = \vec{a}_t = \vec{\alpha} \times \vec{r}$

3-Relation for Time Period:

The time required for one complete revolution or cycle of the motion is called ***"time period"***. It is denoted by "T".

The angular velocity of a particle (body or object) is given by:

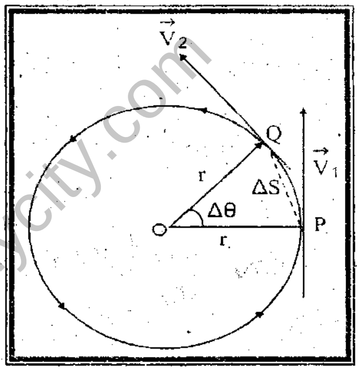
$$\omega = \frac{\theta}{t}$$

If $\theta = 2\pi$ i.e. particle completes one revolution, then

$t = T =$ Time period

$$\omega = \frac{2\pi}{T} \quad \text{OR} \quad \boxed{T = \frac{2\pi}{\omega}}$$

Centripetal Acceleration:



The acceleration of the body directed towards the centre of a circle is called ***"centripetal acceleration"***. Consider an object moving with constant speed "V" in a circle of radius "r".

Let at time "t_1" object is at point P & its linear velocity is $\vec{V}_1$. At a later instant "t_2" velocity is $\vec{V}_2$. The change in velocity "$\Delta\vec{V}$" during time $\Delta t = t_2 - t_1$ is given by:

$\Delta\vec{V} = \vec{V}_2 - \vec{V}_1$ as shown in figure The magnitude of velocity is constant.

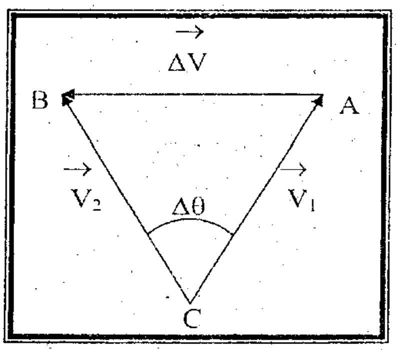
i.e. $V_1 = V_2 = V$

The change in velocity is due to the change in direction of velocity of the object. Since the velocity vectors $\vec{V}_1$ & $\vec{V}_2$ are perpendicular to radii OP & OQ.

$\therefore \quad \angle POQ = \angle ACB = \Delta\theta$

Triangles POQ and ACB are isosceles, therefore their remaining angles P & Q are equal to A & B.

Hence triangles POQ & ACB are similar.



$$\therefore \quad \frac{\Delta V}{V_1} = \frac{\Delta S}{r} \quad \text{But} \quad V_1 = V$$

$$\therefore \quad \frac{\Delta V}{V} = \frac{\Delta S}{r} \quad \text{OR} \quad \Delta V = V\frac{\Delta S}{r}$$

Dividing both sides by Δt

$$\frac{\Delta V}{\Delta t} = \frac{V}{r}\frac{\Delta S}{\Delta t}$$

If Δt approaches to zero then we can write

$$\lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = \frac{V}{r} \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t}$$

But $$\lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = V_{ins} = V$$

And $$\lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = a_{c\,ins} = a_c$$

$$\boxed{a_c = \frac{V^2}{r}} \quad \text{-------------------- (1)}$$

Equation (1) gives the magnitude of the centripetal acceleration while its direction will be same as that of $\Delta\vec{V}$. When Δt approaches to zero $\Delta\theta$ will be very small and $\vec{V}_2$ will be just parallel to $\vec{V}_1$, hence the direction of $\Delta\vec{V}$ will be towards the centre of the circle. Also the direction of centripetal acceleration $\vec{a}_c$ will be towards the centre of the circle.

Centripetal Acceleration in Terms of Angular Velocity and Time Period:

If "ω" be the angular speed, i.e. magnitude of angular velocity of the object then $V = r\omega$ put this value of "V" in equation (1), we get

$$a_c = \frac{r^2\omega^2}{r}$$

OR $\boxed{a_c = r\omega^2}$ -------------------- (2)

If "T" be the time period of the object then

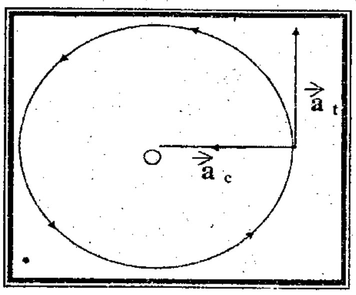
$$T = \frac{2\pi}{\omega} \quad \textbf{OR} \quad \omega = \frac{2\pi}{T}$$

Put this value of "ω" in equation (2), we get

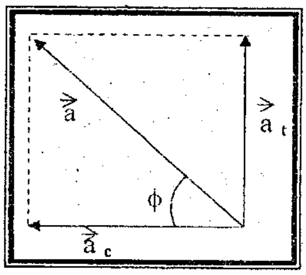
$$\boxed{a_c = \frac{4\pi^2 r}{T^2}}$$

Centripetal and Tangential Acceleration:

The acceleration of an object directed towards the centre of a circle is called ***"Centripetal Acceleration"*** and the acceleration of the object tangent to the curve path is called ***"Tangential Acceleration"***.



Consider an object moving in a circle of radius "r" with variable velocity. The object posseses both centripetal as well as the tangential acceleration. The centripetal acceleration "a_c" is due to the change in direction of the velocity while tangential acceleration "$\vec{a}_t$" is due to the change in magnitude of velocity. Both the accelerations are perpendicular to each other. The resultant are acceleration "$\vec{a}$" can be obtained by using vector diagram $\vec{a} = \vec{a}_c + \vec{a}_t$. The magnitude of $\vec{a}$ is given by $a = \sqrt{a_c^2 + a_t^2}$



The direction of $\vec{a}$ with respect to $\vec{a}_c$ is given by:

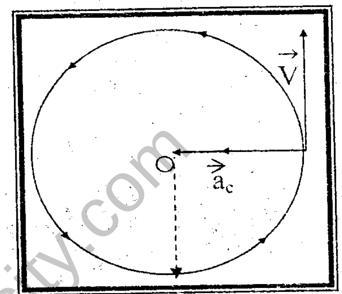
$$\tan\phi = \frac{a_t}{a_c} \quad \textbf{OR} \quad \phi = \tan^{-1}\frac{a_t}{a_c}$$

Centripetal Force:

The force which is directed towards the centre of the circle is called *"Centripetal Force"*.

Consider an object of mass "m" moving in a circle of radius "r" with constant speed "V". The centripetal acceleration "a_c" of the object is given by:

$$a_c = \frac{V^2}{r} \quad \text{------------------------} \quad (1)$$



Since the direction of velocity of the body is changing continuously, therefore, according to the first law of motion, some force is acting on the body continuously. This force is known as ***centripetal force*** and is represented by $\vec{F}_C$.

By Newton's 2nd law of motion, magnitude of the centripetal force is given by:

$$F_C = \frac{mV^2}{r}$$

If "ω" be the angular velocity of the object then:

$$V = r\omega$$

$$\therefore \quad F_C = \frac{mr^2\omega^2}{r}$$

$$\boxed{F_C = mr\omega^2}$$

Centrifugal Force:

The force which is directed away from the centre of the circle is called *"Centrifugal Force"*.

Relation Between Linear And Angular Motion:

	Linear motion (Constant Acceleration. "a")	Rotational motion (Constant Angular Acceleration. "α")
1-	$S = vt$	$\theta = \omega t$
2-	$V_f = V_i + at$	$\omega_f = \omega_i + \alpha t$
3-	$V_{av} = \frac{V_f + V_i}{2}$	$\omega_{av} = \frac{\omega_f + \omega_i}{2}$
4-	$S = V_i t + \frac{1}{2}at^2$	$\theta = \omega_i t + \frac{1}{2}\alpha t^2$
5-	$2aS = V_f^2 - V_i^2$	$2\alpha\theta = \omega_f^2 - \omega_i^2$

Q. SHOW THAT $R_{max} = 4H_{max}$

As $H_{max} = \frac{V_0^2 Sin^2\theta}{2g}$

For maximum range

$\theta = 45^0$

$$H_{max} = \frac{V_0^2 (Sin\,45^0)^2}{2g}$$

$$H_{max} = \frac{V_0^2 (\frac{1}{\sqrt{2}})^2}{2g} \qquad \therefore Sin\,45^0 = 0.707 = \frac{1}{\sqrt{2}}$$

$$H_{max} = \frac{V_0 \times \frac{1}{2}}{2g}$$

$$H_{max} = \frac{V_0^2}{4g}$$

$$4H_{max} = \frac{V_0^2}{g} \qquad \therefore R_{max} = \frac{V_0^2}{g}$$

$$4H_{max} = R_{max}$$

$\boxed{R_{max} = 4H_{max}}$ Proved

Q. SHOW THAT THE RANGE OF PROJECTILE IS THE SAME FOR

$\theta = 45^0 + \alpha$ and $\theta' = 45^0 - \alpha$

where $\alpha < 45^0$

As $R = \frac{V_0^2 Sin\,2\theta}{g}$

$\theta = 45^0 + \alpha$

$$R = \frac{V_0^2 Sin\,2(45^0 + \alpha)}{g}$$

$$R = \frac{V_0^2 Sin\,(90^0 + 2\alpha)}{g}$$

$\therefore Sin\,(90^0 + 2\alpha) = Cos\,2\alpha$

$$R = \frac{V_0^2 Sin\,Cos2\alpha}{g} \rightarrow (1)$$

$R = \frac{V_0^2 Sin\,2\theta}{g}$

$\theta' = 45^0 - \alpha$

$$R' = \frac{V_0^2 Sin\,2(45^0 - \alpha)}{g}$$

$$R' = \frac{V_0^2 Sin\,(90^0 \quad 2\alpha)}{g}$$

$\therefore Sin\,(90^0 - 2\alpha) = Cos\,2\alpha$

$$R' = \frac{V_0^2 Sin\,Cos2\alpha}{g} \rightarrow (2)$$

Comparing equation (1) and equation (2)

$\boxed{R = R'}$