PHYSICAL QUANTITY

1. Definition:- The quantity which can be measured is known as Physical Quantity.

OR

The property of the matter which is related to its measurement is called Physical Quantity.

2. Types:- There are two types of the physical quantities.

(a)- Scalar Quantity

(b)- Vector Quantity.

SCALAR

1- Definition:- It is a physical quantity which is completely described by its magnitude with proper unit.

2. Examples:-

Mass, distance, speed, energy, work volume, temperature, time, area, potential power, electric charge etc.

The scalars are added, subtracted, multiplied and divided by ordinary rules of arithmetic.

VECTOR

1. Definition:- A physical quantity which is completely described by its magnitude with proper unit as well as direction.

2- Examples:-

Displacement, velocity, acceleration, force weight, momentum, torque, angular velocity angular acceleration, etc.

3- Vectors cannot be added, subtracted, multiplied and divided by ordinary arithmetic rules but we use the methods of vector addition

VECTOR REPRESENTATION

1- Symbolically / Mathematically :-

- A vector is represented by a letter with an arrow drawn above or below it.

e.g :- $\vec{A}$, $\underrightarrow{A}$, $\vec{a}$, $\underrightarrow{a}$.

- A vector can also be represented by a bold face letter.

e.g :- **A** , **a** , **b**

Magnitude of a vector

The magnitude of a vector is called modulus or simply mod. of a vector.

- It is represented by a vector between two small parallel lines

i-e

$|\vec{A}|$ = Modulus of vector $\vec{A}$

= Magnitude of vector $\vec{A}$.

- The magnitude of a vector can also be represented by an italic letter " *A* "

2. Graphically :-

Graphically a vector is represented by a straight line which is drawn according to the selected scale with an arrow head. The length of the line gives the magnitude of the vector and arrow head indicates the direction of the vector.

In order to represent a certain vector we require two things.

(i)- A suitable scale

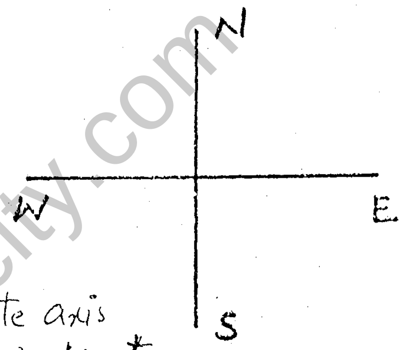
(ii)- Direction Indicator (In case of geographic dir.)

- Reference axes (In case of angular direction).

(i)- **Suitable Scale** :- A suitable scale is selected so that vector should have length according to the magnitude of the vector.

(ii). **Direction Indicator** :-

Direction indicator consists of two mutually perpendicular lines. The horizontal line indicates East - West and vertical line indicates North - South. By comparing the arrow-head of a vector with direction indicator, we can say about its direction.

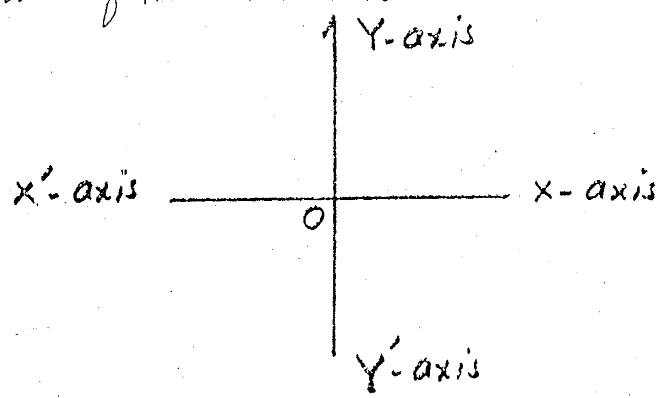


⇒ For angular direction reference axes or coordinate axis are used instead of direction indicator.

Reference Axes / Coordinate Axes.

In 2 - Dimension

Two lines drawn at right angles to each other are known as coordinate axes and their point of intersection is known as origin.

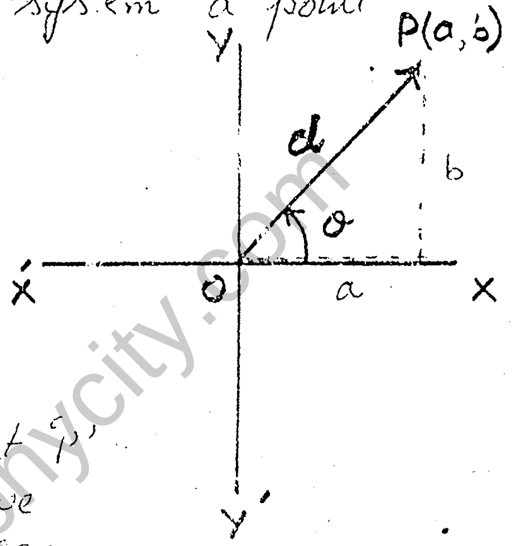


One of the lines is named as x-axis, and the other y-axis. Usually the x-axis is taken as the horizontal axis, with the positive direction to the right, and y-axis as the vertical axis with the positive direction upward.

Cartesian or Rectangular Coordinate system

"The system of coordinate axes in which we use x and y-components of a vector is called Cartesian or Rectangular coordinate system."

In cartesian coordinate system a point 'P' in a plane having coordinates (a, b) can be represented by a representative line $\overrightarrow{OP}$ making an angle 'θ' with +ve x-axis in anticlockwise direction as shown in figure.

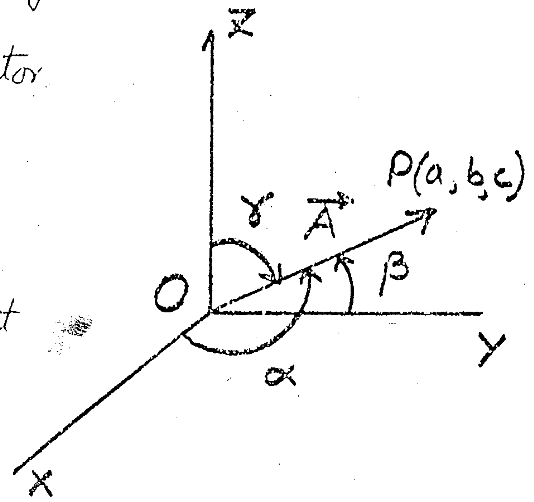


The co-ordinates of point 'P' (a, b) means that if we start from origin, we can reach 'P' by moving 'a' unit along +ve x-axis and then 'b' unit along the +ve y-axis.

In 3-Dimensions

The direction of a vector in space requires another axis which is at right angle to both x and y-axes. The third axis is called Z-axis.

The direction of a vector in space is specified by three angles which the representative line of the vector makes with x, y and z-axes respectively. The point 'P' of a vector $\vec{A}$ is thus denoted by three coordinates (a, b, c).



ADDITION OF VECTORS BY GRAPHICAL METHOD (5)

RESULTANT VECTOR :- The resultant (or sum) of two or more vectors is a single vector which has the same effect as the combined effect of all the vectors to be added. The single vector is known as the resultant vector.

➔ If $\vec{A_1}, \vec{A_2}, \dots\dots, \vec{A_n}$ are 'n' vectors, then their resultant is symbolically represented by $\vec{A}$ as

$$\vec{A} = \vec{A_1} + \vec{A_2} + \dots\dots + \vec{A_n} \qquad (1)$$

or $$\vec{A} = \sum_{r=1}^{n} \vec{A_r}$$

Symbol 'Σ' (sigma) stands for summation. The above equation does not stands for arithmatic addition. Vectors may be added by graphical method or by trignometric method.

— Graphically vectors are added by Head To Tail Rule

HEAD TO TAIL RULE

This rule is described as:

Step - 1 :- First of all, representative lines of all the vectors to be added are drawn with respect to suitable reference axis by choosing a proper scale. Head and tail of each vector is indicated clearly.

Step - 2 :- Redraw the representative lines of the vectors in such a way that the head of one vector coincides with tail of the other.

Step - 3 :- The resultant of these vectors is obtained by the joining the tail of first vector with the head of the last one.

Explanation :-

Consider two vectors $\vec{A}$ and $\vec{B}$ represented by the straight lines $\vec{OP}$ and $\vec{OQ}$ respectively, according to a suitable scale as shown in figure (a).

Firstly vector $\vec{A}$ is redrawn in a certain frame of reference. Vector $\vec{B}$ is drawn in such a way that tail of second vector coincides with head of the first vector as shown in figure (b).

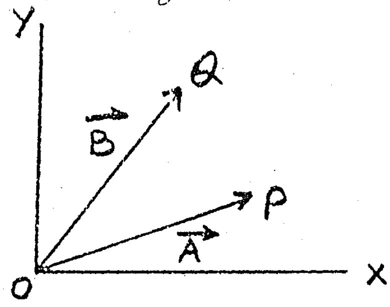


Fig (a)

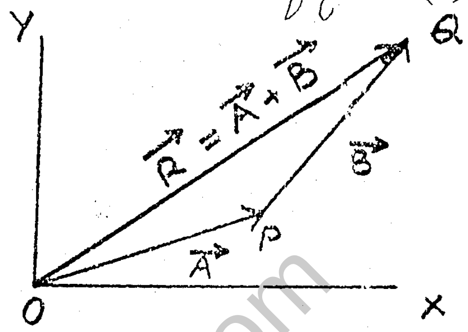


Fig. (b)

Let $\vec{R}$ be the resultant vector which is represented by the line $\overrightarrow{OQ}$. i-e

$$\vec{R} = \vec{A} + \vec{B} \quad \text{———} \quad (2)$$

Secondly we join tail of $\vec{A}$ with the head of vector $\vec{B}$.

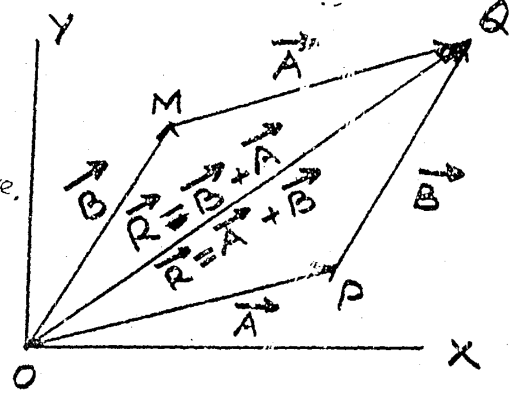
In this case

$$\vec{R} = \vec{B} + \vec{A} \quad \text{———} \quad (3)$$

Therefore

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

So the vector addition is said to be Commutative. It means that when vectors are added, the resultant is same for any order of addition.

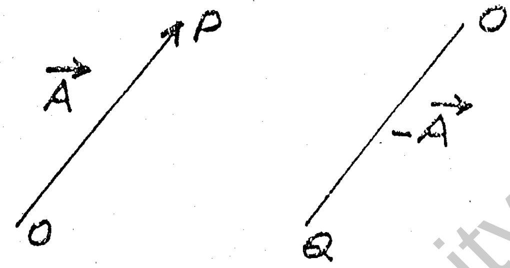


NOTE:- If vectors are parallel to each other then by head-to-tail rule, the magnitude of the resultant vector is equal to the sum of the magnitude of all the vectors which are to be added.

NEGATIVE OF A VECTOR

1- **Definition :-** A vector having the same magnitude as that of the given vector but opposite in direction is called negative of a vector.

2- **For Example :-** If $\vec{A}$ is a given vector represented by a line $\vec{OP}$, then its negative will be $-\vec{A}$ represented by the line $\vec{OQ}$.



$|\vec{OP}| = |\vec{OQ}|$

-ve sign indicates the opposite direction.

VECTOR SUBTRACTION

1- **Definition :-** The subtraction of two vectors gives a resultant vector which is the addition of one vector with negative of the other one.

2- **Explanation :-**

The subtraction of a vector is equivalent to the addition of same vector with its direction reversed.

Thus to subtract vector $\vec{B}$ from vector $\vec{A}$, the following steps are applied:

(a) - Draw the representative lines of vectors $\vec{A}$ and $\vec{B}$ in a certain frame of reference.

(b) - Reverse the direction of vector $\vec{B}$ which is to be subtracted from vector $\vec{A}$. (i-e draw the negative vector of $\vec{B}$). as shown in Fig (b)

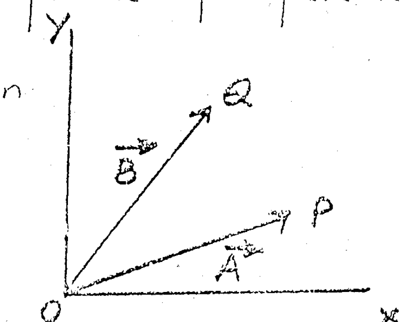


Fig (a)

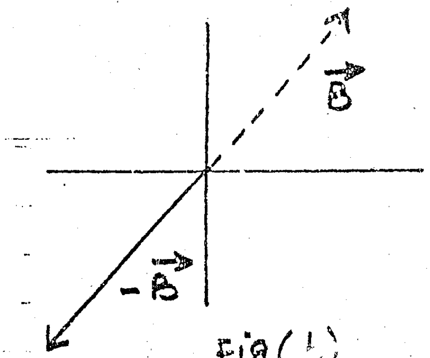


Fig (b)

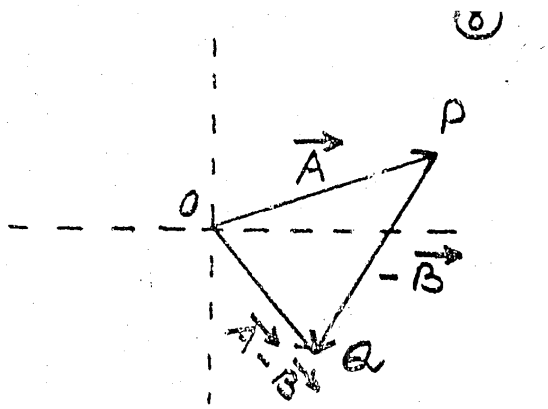


Fig (c)

(c). Add $-\vec{B}$ to the first vector $\vec{A}$ by head-to-tail rule as shown in fig (c).
i-e

$$\vec{A} + (-\vec{B}) = \vec{A} - \vec{B}$$

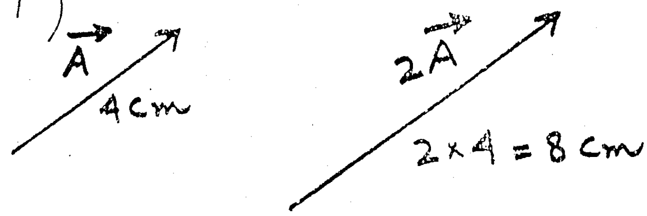
MULTIPLICATION OF A VECTOR BY A SCALAR

1- Multiplication with numbers

(a)- A Positive number:-

When a vector $\vec{A}$ is multiplied by a positive number $n > 0$, then the product is defined to be a new vector $n\vec{A}$ having the same direction as $\vec{A}$ but a magnitude n times to the magnitude of $\vec{A}$. (i-e $n|\vec{A}|$)

e.g :-

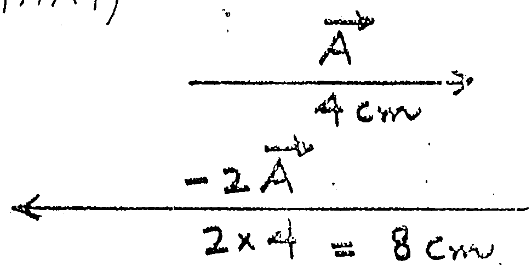


(b) A Negative number:-

When a vector $\vec{A}$ is multiplied by a negative number "$-n$", then the product is defined to be a new vector $-n\vec{A}$

having the opposite (reverse) direction as $\vec{A}$ but a magnitude n times to the magnitude of $\vec{A}$. (i-e $n|\vec{A}|$ or $|n\vec{A}|$)

e.g :-



- Magnitude of a vector cannot be negative. Negative sign just indicates its opposite direction.

2- Multiplication with Scalar quantity :-

When 'n' represents a physical quantity with dimension, the product $n\vec{A}$ will correspond to a new physical quantity (i-e a vector).
This resulting vector will have the same dimensions as the product of the dimensions of two quantities which were multiplied together.

e.g :- When mass 'm' is multiplied by velocity $\vec{V}$ the product is a new vector quantity called momentum $\vec{P}$ having the same dimensions as those of mass and velocity

$$m\vec{V} = \vec{P}$$

$$Kg\,m\,s^{-1} = Kg\,m\,s^{-1}$$

$$[MLT^{-1}] = [MLT^{-1}]$$

UNIT VECTOR

1- **Definition :-** A vector whose magnitude is one and it points in the direction of a given vector is called a unit vector.

2- **Explanation :-** (a) It is used to represent the direction of a vector. It is represented symbolically by a letter with a hat "^" on it.

i.e $\hat{A}$ (it can be read as "A hat").

Now consider a vector $\vec{A}$ having magnitude $|\vec{A}|$ and its unit vector is $\hat{A}$. Then vector $\vec{A}$ can be determined as

$$\vec{A} = |\vec{A}|\hat{A} = A\hat{A}$$

Therefore a vector can be obtained by multiplying its magnitude with unit vector.

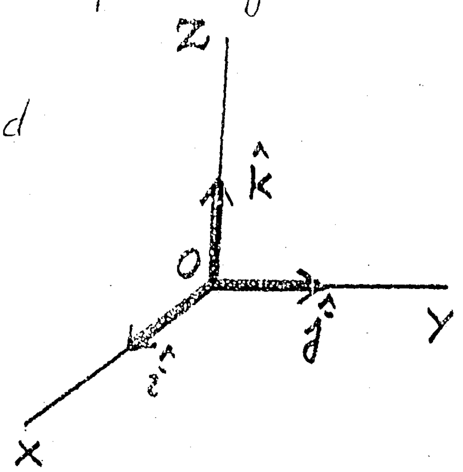
From above equation unit vector can be obtained as

$$\boxed{\hat{A} = \frac{\vec{A}}{A}}$$

(b) Unit Orthogonal Vectors

The direction along x, y and z-axis are generally represented by unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ respectively. Since these unit vectors are mutually perpendicular to each other so these are called Unit Orthogonal vectors.

$$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$$



Component of a vector :-

A component of a vector is its effective value or projection in a given direction.
A vector may be considered as the resultant of its component vectors along the specified direction.

RECTANGULAR COMPONENTS OF A VECTOR

Def:- The components of a vector which are at right angles to eachother are called rectangular components.

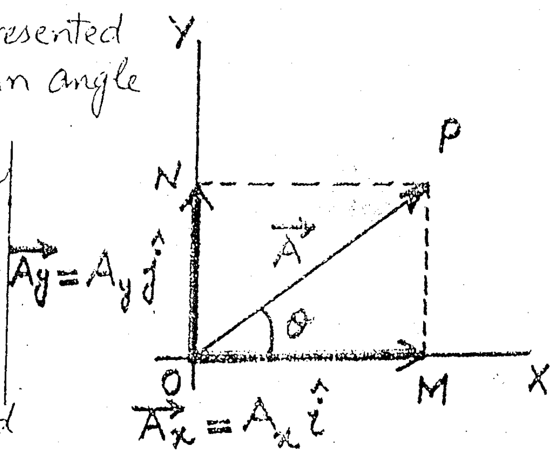
OR

The components of a vector which are along mutually perpendicular directions are called rectangular components.

- The component which is along x-axis is called horizontal or x-component, while the other which is along y-axis is called vertical component or y-component.

Determination of Rectangular components

Consider a vector $\vec{A}$ represented by the line OP making an angle 'θ' with x-axis.
Draw perpendicular from point 'P' on x-axis, then OM will be the effective value or projection of vector $\vec{A}$ along x-axis, and it is represented by $\vec{A_x}$ or $A_x\hat{i}$.



Now draw perpendicular from point 'P' on y-axis, then PM will be the effective value or projection of vector $\vec{A}$ along y-axis and it is represented by $\vec{A_y}$ or $A_y\hat{j}$.
Adding $\vec{A_x}$ and $\vec{A_y}$ by head to tail rule

$$\vec{A} = \vec{A_x} + \vec{A_y}$$

or $$\vec{A} = A_x\hat{i} + A_y\hat{j} \quad \text{———} \quad (1)$$

Thus $A_x\hat{i}$ and $A_y\hat{j}$ are the components of vector $\vec{A}$. Since these are at right angle to each other, hence these are called rectangular components of $\vec{A}$.

A_x To find out the value of x-component consider right angled triangle OPM

$$\cos\theta = \frac{\text{Base}}{\text{Hyp}} = \frac{OM}{OP} = \frac{A_x}{A}$$

$$\therefore \quad A_x = A\cos\theta$$

and $$A_x\hat{i} = (A\cos\theta)\,\hat{i}$$

A_y

$$\sin\theta = \frac{\text{Perp}}{\text{Hyp}} = \frac{PM}{OP} = \frac{A_y}{A}$$

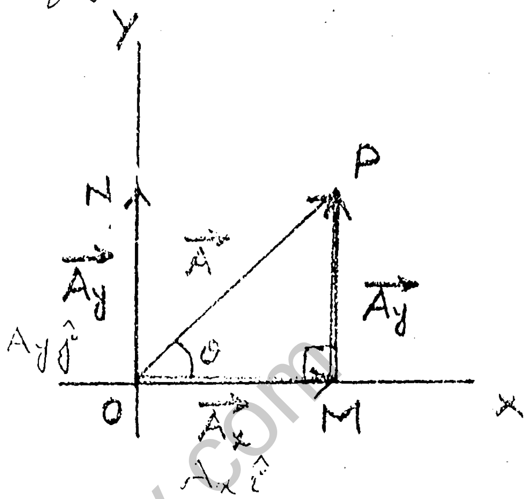
$$\therefore \quad A_y = A\sin\theta$$

$$\text{and} \quad A_y\hat{j} = (A\sin\theta)\,\hat{j}$$

Putting values of A_x and A_y in equation (1)
So $$\vec{A} = (A\cos\theta)\,\hat{i} + (A\sin\theta)\,\hat{j}.$$

Determination of a Vector from its Rectangular Components:-

If the rectangular components of a vector are given. Then adding both component vectors by head to tail rule as shown in figure.



We obtain $\vec{A}$ as a resultant vector of the components.

i-e

$$\vec{A} = \vec{A_x} + \vec{A_y}$$

or $\vec{A} = A_x\hat{i} + A_y\hat{j}$

For MAGNITUDE

We can find out the magnitude of vector by using Pythagorean Theorm on rt ∠d △ OPM.

$$(OP)^2 = (OM)^2 + (MP)^2$$

$$A^2 = A_x^2 + A_y^2$$

So $$\boxed{A = \sqrt{A_x^2 + A_y^2}}$$

For Direction:-

Applying trignometric ratio on the rt ∠d △ OPM

$$\tan\theta = \frac{MP}{OM} = \frac{A_y}{A_x}$$

$$\boxed{\theta = \tan^{-1}\frac{A_y}{A_x}}$$

Therefore a vector $\vec{A}$ has been completely determined by finding its magnitude and direction.

POSITION VECTOR

1- **Definition :-** A vector which describes the location of a particle with respect to the origin is called position vector.

2. **Explanation :-**

(a)- The position vector is denoted by $\vec{r}$. It is represented by a straight line drawn in such a way that its tail coincides with the origin and head with the point at which particle is located.

(b)- **In 2-Dimension :-**

In order to locate a particle in a plane (in 2-dimension) a vector is drawn from origin 'O' to 'P'. The projection of position vector $\vec{r}$ on the x and y-axes are the coordinates 'a' and 'b' and they are the rectangular components of the vector $\vec{r}$. Hence

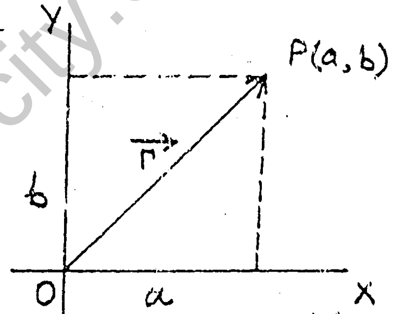


fig.(a)

$$\vec{r} = a\hat{i} + b\hat{j}$$

and

$$r = \sqrt{a^2 + b^2}$$

(c)- **In 3-Dimension :-**

In three dimensional space the position vector $\vec{r}$ of a point P(a, b, c) is shown in figure (b). It can be represented as

$$\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$$

and

$$r = \sqrt{a^2 + b^2 + c^2}$$

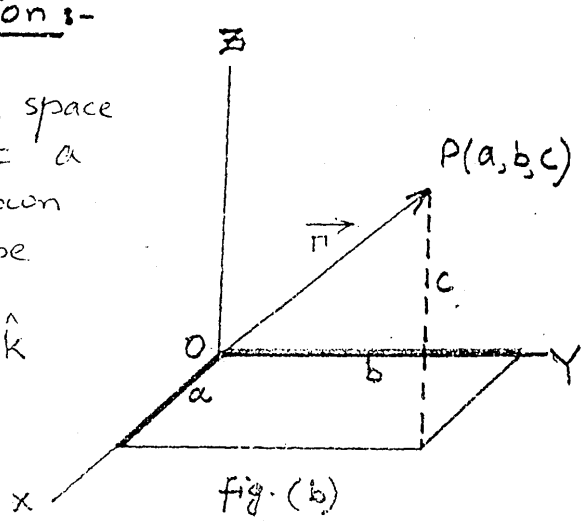


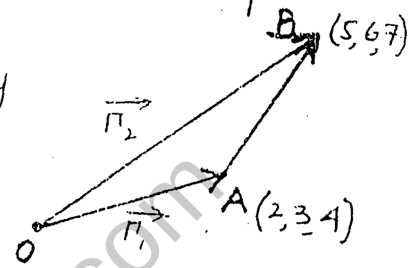
fig.(b)

Example 2.1:- The position of two aeroplanes at any instant are represented by two points A(2,3,4) and B(5,6,7) from an origin 'O' in Km.

(i)- What are their position vectors?

(ii)- Calculate the distance between the two aeroplanes.

Solution:-



(i)- A position vector is given by

$$\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$$

Thus position vector of first aeroplane 'A' is

$$\vec{r_1} = \vec{OA} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$

and the second aeroplane 'B' is

$$\vec{r_2} = \vec{OB} = 5\hat{i} + 6\hat{j} + 7\hat{k}$$

As from figure by head to tail rule

$$\vec{OA} + \vec{AB} = \vec{OB}$$

There-fore the displacement vector between two aeroplanes is given by

$$\vec{AB} = \vec{OB} - \vec{OA}$$

i.e $\vec{AB} = (5\hat{i} + 6\hat{j} + 7\hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k})$

$$= 5\hat{i} - 2\hat{i} + 6\hat{j} - 3\hat{j} + 7\hat{k} - 4\hat{k}$$

$$= (5-2)\hat{i} + (6-3)\hat{j} + (7-4)\hat{k}$$

$$\vec{AB} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

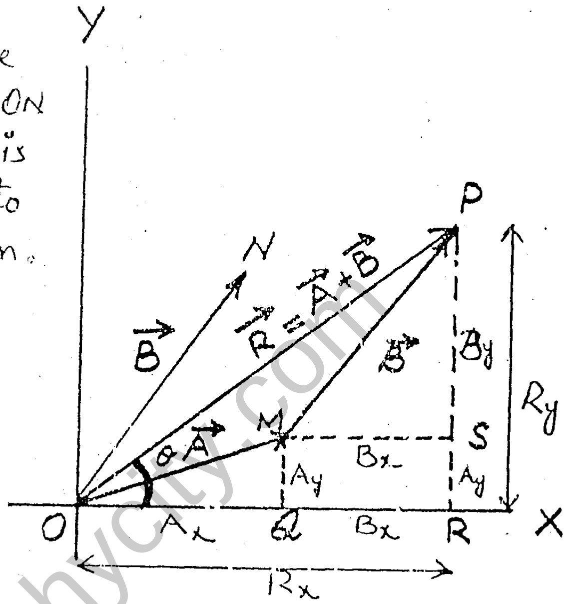
Thus magnitude of vector $\vec{AB}$ is the distance between the position of two aeroplanes which we have

$$|\vec{AB}| = \sqrt{(3)^2 + (3)^2 + (3)^2} = \sqrt{27}$$

$$|\vec{AB}| = 5.2\ Km$$

VECTOR ADDITION BY RECTANGULAR COMPONENTS

Let $\vec{A}$ and $\vec{B}$ be two vectors which are represented by the two directed lines OM and ON respectively. The vector $\vec{B}$ is added to $\vec{A}$ by the head to tail rule of vector addition. Let $\vec{R}$ be the resultant vector which is represented by the line OP which is making an angle 'θ' with x-axis.



$$\vec{R} = \vec{A} + \vec{B}$$

In order to resolve the vectors $\vec{R}$, $\vec{A}$ and $\vec{B}$ into rectangular components, draw perpendiculars MQ and PR from points 'M' and 'P' on x-axis. Also draw perpendicular from M on the line PR.

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j}$$
and $$\vec{R} = R_x\hat{i} + R_y\hat{j}.$$

As from figure

$OQ = A_x$, $QM = A_y$
$MS = B_x$, $SP = B_y$
$OR = R_x$ $RP = R_y$

From geometery of the figure

- Along x-axis.

$$OR = OQ + QR$$
Since $QR = MS$
$$\therefore \quad OR = OQ + MS$$

$$\Rightarrow \boxed{R_x = A_x + B_x} \quad \text{——— (1)}$$

or $\vec{R_x} = R_x\hat{i} = (A_x + B_x)\hat{i}$

which means that the sum of the magnitudeas of x-components of two vectors which are to be added is equal to the x-component of the resultant.

Again from geometery of fig. along Y-axis

$PR = RS + SP$

As $RS = QM$

So $PR = QM + SP$

$$\Rightarrow \boxed{R_y = A_y + B_y} \quad \text{——— (2)}$$

or $\vec{R_y} = R_y\hat{j} = (A_y + B_y)\hat{j}$

The eq (2) shows that the sum of the magnitudes of y-components of two vectors which are to be added is equal to the y-component of the resultant.

Since $R_x\hat{i}$ and $R_y\hat{j}$ are the rectangular components of Resultant vector $\vec{R}$.

Hence

$$\boxed{\vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}}$$

MAGNITUDE:-

As $\vec{R} = R_x\hat{i} + R_y\hat{j}$

The magnitude of the resultant vector $\vec{R}$ is given as

$$R = \sqrt{R_x^2 + R_y^2}$$

$$\therefore \boxed{R = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2}}$$

DIRECTION :- Direction of the resultant vector $\vec{R}$ can be determined as

$$\theta = \tan^{-1} R_y/R_x$$

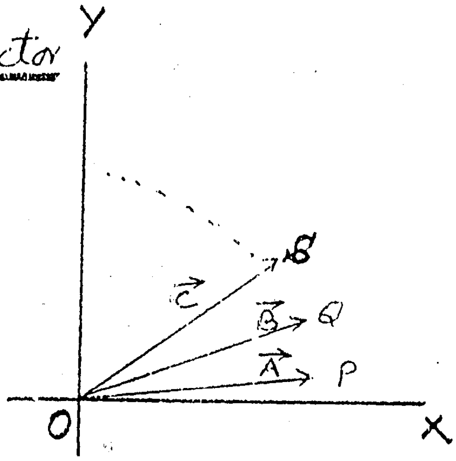
$$\boxed{\theta = \tan^{-1}\left(\frac{A_y+B_y}{A_x+B_x}\right)}$$

FOR ANY NUMBER OF COPLANER VECTORS

COPLANER VECTORS

The vectors which are lying in the same plane are known as coplaner vectors.

Explanation for Resultant Vector



Let we have any number of coplaner vectors $\vec{A}, \vec{B}, \vec{C}, \ldots$

Let $\vec{R}$ be their resultant vector i-e

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \cdots$$

and $\vec{R} = R_x\hat{i} + R_y\hat{j}$

Then according to equation ① and ②

$$R_x = A_x + B_x + C_x + \cdots$$

and

$$R_y = A_y + B_y + C_y + \cdots$$

FOR MAGNITUDE :- Using formula for magnitude of a vector.

$$R = \sqrt{R_x^2 + R_y^2}$$

or

$$R = \sqrt{(A_x + B_x + C_x + \cdots)^2 + (A_y + B_y + C_y + \cdots)^2}$$

FOR DIRECTION :- Using formula for direction.

$$\theta = \tan^{-1} R_y/R_x .$$

$$\theta = \tan^{-1}\left(\frac{A_y + B_y + C_y + \cdots}{A_x + B_x + C_x + \cdots}\right)$$

MAIN STEPS

The vector addition by rectangular components consists of the following steps.

Step-1 :- Find 'x' and 'y' components of all the given vectors.

Step-2 :- Find x-component 'R_x' of the resultant vector by adding the x-comp. of all vectors.

Step-3 :- Find y-component 'R_y' of the resultant vector by adding the y-comp. of all vectors.

Step-4 :- Find the magnitude of resultant vector $\vec{R}$ by using

$$R = \sqrt{R_x^2 + R_y^2}$$

Step-5 :- Find the direction of resultant vector $\vec{R}$ by using

$$\theta = \tan^{-1}\frac{R_y}{R_x}$$

where 'θ' is the angle which resultant vector makes with +ve x-axis.

EXISTANCE OF THE RESULTANT VECTOR IN QUADRANT

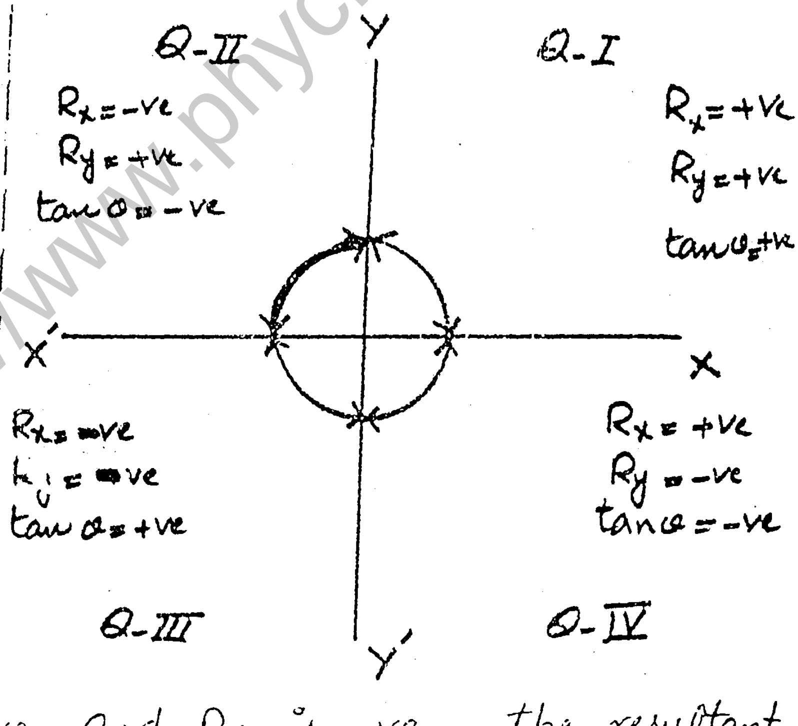
The signs of 'R_x' and 'R_y' determines the quadrant in which resultant vector lies. For that purpose proceed as given below.

1)- Determine the value of $\tan^{-1} R_y/R_x = \phi$ from the calculator or by consulting trignometric tables by ignoring the sign of 'R_x' and 'R_y'.

2)- If both R_x and R_y components are positive, then, the resultant lies in the first quadrant and its direction is $\theta = \tan^{-1}\frac{R_y}{R_x} = \phi$.

3)- If R_x is $-$ve and R_y is $+$ve the resultant lies in the second quadrant and its direction is $\theta = 180° - \phi$.

4)- If both R_x and R_y are $-$ve, the resultant lies in the third quadrant and its direction is $\theta = 180 + \phi$



5)- If R_x is $+$ve and R_y is $-$ve, the resultant lies in the fourth quadrant and its direction is

$$\theta = 360° - \phi$$

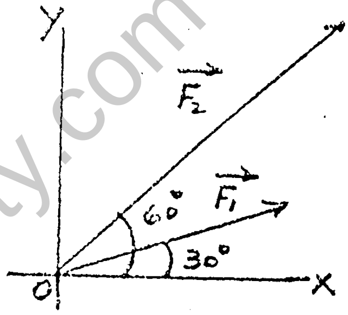
NOTE:- Consider two vectors $\vec{A}$ and $\vec{B}$. Let θ be the angle between them, then their resultant vector will have magnitude $R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$.

Example 2.2:- Two forces of magnitude 10N and 20N act on a body in directions making angle 30° and 60° with x-axis respectively. Find the resultant force.

Solution:- $F_1 = 10N$ $\theta_1 = 30°$ $F_2 = 20N$ $\theta_2 = 60°$

Step (i)-

X - Components



The x-component of first force

$$F_{1x} = F_1 \cos\theta_1$$
$$= (10N)\cos 30°$$
$$= 10 \times 0.866$$
$$F_{1x} = 8.66N$$

The x-component of second force

$$F_{2x} = F_2 \cos\theta_2 = (20\cos 60°)N$$
$$F_{2x} = 20 \times 0.5 = 10N$$

Y- Components

The y-component of first force

$$F_{1y} = F_1 \sin\theta_1 = (10\sin 30°)N$$
$$F_{1y} = 10 \times 0.5 = 5N$$

The y-component of second force

$$F_{2y} = F_2 \sin\theta_2 = (20\sin 60°)N$$
$$= (20 \times 0.866)N$$
$$F_{2y} = 17.32N$$

Step-(ii)- The magnitude of x-component F_x of the resultant force $\vec{F}$.

$$F_x = F_{1x} + F_{2x}$$
$$F_x = 8.66 + 10 = 18.66N$$

Step - III

The magnitude of y-components F_y of $\vec{F}$

$$F_y = F_{1y} + F_{2y}$$
$$= 5 + 17.32 = 22.32\,N$$

Step - IV

The magnitude 'F' of the resultant force $\vec{F}$

$$F = \sqrt{F_x^2 + F_y^2}$$
$$= \sqrt{(18.66)^2 + (22.32)^2} = 29\,N$$
$$\therefore \boxed{F = 29N}$$

Step - V

If the resultant force $\vec{F}$ makes an angle 'θ' with x-axis, then

$$\theta = \tan^{-1}\frac{F_y}{F_x} = \tan^{-1}\frac{22.32N}{18.66N}$$

$$\theta = \tan^{-1}(1.196)$$
$$\boxed{\theta = 50^\circ}$$

EXAMPLE 2.3

Find the angle between two forces of equal magnitude when the magnitude of their resultant is also equal to the magnitude of either of these forces.

Solution :-

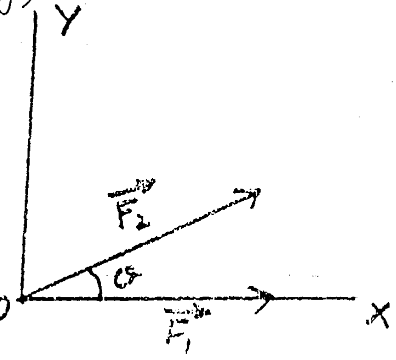
$$|\vec{F_1}| = |\vec{F_2}| = F \text{ (Say)}$$

and $\quad |\vec{R}| = F$

Let $\vec{F_1}$ be lying on x-axis

So $\theta_1 = 0^\circ$

and $\vec{F_2}$ be making an angle 'θ' with $\vec{F_1}$ or with x-axis.



Step - I   x - components

$$F_{1x} = F_1 \cos\theta_1 = F\cos 0 = F$$

$F_{2x} = F_2 \cos\theta = F\cos\theta$

Y- Components

$F_{1y} = F_1 \sin\theta_1 = F\sin 0 = \text{Zero}$

$F_{2y} = F_2 \sin\theta = F\sin\theta$

Step - II The magnitude of x-component 'R_x' of the resultant vector $\vec{R}$.

$R_x = F_{1x} + F_{2x}$

$R_x = F + F\cos\theta$

Step - III

The magnitude of y-component 'R_y' of the resultant vector $\vec{R}$

$R_y = F_{1y} + F_{2y}$

$= 0 + F\sin\theta = F\sin\theta$

Step - IV

The magnitude of resultant vector $\vec{R}$

$R = \sqrt{R_x^2 + R_y^2}$

As $R = F$

$F = \sqrt{(F + F\cos\theta)^2 + (F\sin\theta)^2}$

$F = \sqrt{F^2 + F^2\cos^2\theta + 2F^2\cos\theta + F^2\sin^2\theta}$

$= \sqrt{F^2 + F^2(\cos^2\theta + \sin^2\theta) + 2F^2\cos\theta}$

$F = \sqrt{2F^2 + 2F^2\cos\theta}$

Squaring on both sides

$F^2 = 2F^2 + 2F^2\cos\theta$

or

$2F^2\cos\theta = F^2 - 2F^2$

$2F^2\cos\theta = -F^2$

$\cos\theta = -\frac{1}{2}$

$\theta = \cos^{-1}(-0.5)$

So $\theta = 120^\circ$

PRODUCT OF TWO VECTORS

There are two types of vector multiplications.

1. SCALAR PRODUCT
2. VECTOR PRODUCT

1. SCALAR PRODUCT

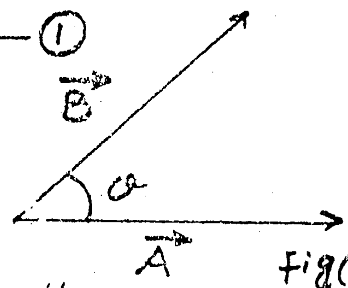
(a)- Definition :- When the product of two vectors results into a scalar quantity, the product is called Scalar Product.

(b). Explanation :-

(i). The scalar product is denoted by putting a dot ($\cdot$) between the two vectors. The scalar product of two vectors $\vec{A}$ and $\vec{B}$ is written as $\vec{A}\cdot\vec{B}$. Hence scalar product is also known as DOT PRODUCT.

(ii). Formula :- If there are two vectors $\vec{A}$ and $\vec{B}$ making an angle 'θ' with each other, then scalar or dot product is given as

$$\boxed{\vec{A}\cdot\vec{B} = AB\cos\theta} \quad \text{—— (1)}$$



fig(a)

where A and B are the magnitudes of vectors $\vec{A}$ and $\vec{B}$ respectively, and '$\cos\theta$' is a pure number (i-e scalar), therefore scalar product is a scalar quantity.

(iii)- Another Definition :-

Let vectors $\vec{A}$ and $\vec{B}$ be represented by the lines $\vec{OP}$ and $\vec{OQ}$ respectively as shown in fig (b). Then

$$\vec{A}\cdot\vec{B} = AB\cos\theta = A(B\cos\theta) \quad \text{—— (2)}$$

Draw perpendicular from point 'Q' on $\vec{OP}$.

$OM = B\cos\theta$ = Magnitude of component of $\vec{B}$ along $\vec{A}$.

From eq (2)

$\vec{A}\cdot\vec{B}$ = (Magnitude of $\vec{A}$)(Magnitude of comp. of $\vec{B}$ in the direction of $\vec{A}$) —— (3)

or

$\vec{A}\cdot\vec{B}$ = (Magnitude of $\vec{A}$)(Projection of $\vec{B}$ on $\vec{A}$).

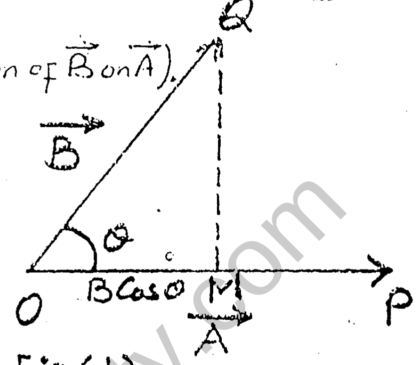


Fig (b)

Now draw perpendicular from point 'P' on OQ as shown in figure (c)

Thus

$ON = A\cos\theta$

= Magnitude of component of $\vec{A}$ on $\vec{B}$

$\vec{B}\cdot\vec{A} = BA\cos\theta$

$= B(A\cos\theta)$ —— (4)

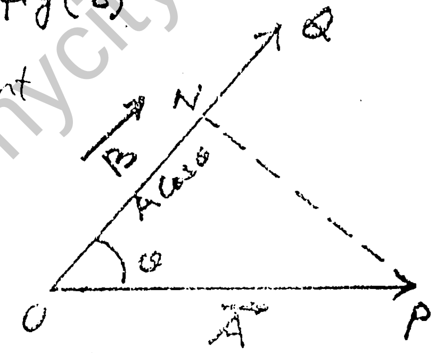


Fig.(c)

$\vec{B}\cdot\vec{A}$ = (Magnitude of $\vec{B}$)(Magnitude of comp. of $\vec{A}$ along $\vec{B}$) (5)

or

$\vec{B}\cdot\vec{A}$ = (Magnitude of $\vec{B}$)(Projection of $\vec{A}$ on $\vec{B}$).

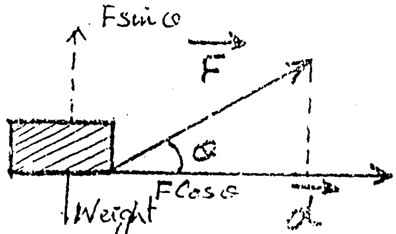
Hence eq (3) and (5) give us another definition of scalar product.

"The scalar product of the two vectors is the multiplication of the magnitude of first vector with the magnitude of the component of second in the direction of first vector."

NOTE :- Inner Product :- This product is also known as inner product because both the vectors $\vec{A}$, $\vec{B}$ and result of their dot product $\vec{A}\cdot\vec{B} = AB\cos\theta$ also lying in the same plane.

(iv). EXAMPLES

(A). Work :-



Fig(d)

Consider the work done by a force $\vec{F}$ whose point of application moves a distance d in the direction making an angle "θ" with the line of action of $\vec{F}$ as shown in figure (d). F Sin θ is balanced by weight of the body.

Work done = (effective component of force in the direction of motion)(distance covered)

$$= (F\cos\theta)d$$

$$= Fd\cos\theta$$

$$\therefore \quad Work = \vec{F}\cdot\vec{d}$$

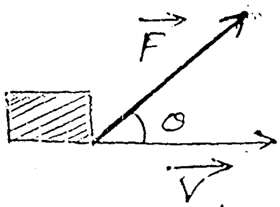
So work is dot product of force and displacement.

(B). Power :-

$$Power = \frac{Work}{Time} = \frac{\vec{F}\cdot\vec{d}}{t}$$

$$Power = \vec{F}\cdot\left(\frac{\vec{d}}{t}\right)$$

$$Power = \vec{F}\cdot\vec{V}$$



or $\quad Power = FV\cos\theta$

$\therefore$ Power is the scalar product of force and velocity.

(C). Electric Flux :-

It is the dot product of electric intensity and vector area.

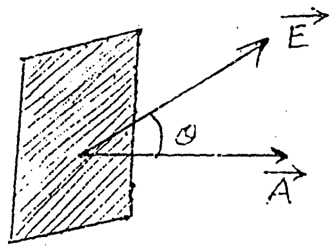
$$\phi_e = \vec{E}\cdot\vec{A}$$

$$= EA\cos\theta$$

where

$\vec{E}$ = Electric Intensity

$\vec{A}$ = vector area



(V). CHARACTERISTICS OF SCALAR PRODUCT.

A. Commutative Law:-

As

$$\vec{A} \cdot \vec{B} = AB\cos\theta \quad \text{———} \quad (1)$$

And $\vec{B} \cdot \vec{A} = BA\cos\theta$

or $\vec{B} \cdot \vec{A} = AB\cos\theta \quad \text{———} \quad (2)$

Comparing eq (1) and (2)

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

This shows that the order of multiplication is irrelevant. Hence scalar product is commutative.

B. Associative Law:-

Dot product obeys associative law

$$m(\vec{A}) \cdot n(\vec{B}) = mn(\vec{A} \cdot \vec{B}) = mn\vec{A} \cdot \vec{B}$$
$$= \vec{A} \cdot mn\vec{B}$$
$$= n\vec{A} \cdot m\vec{B}$$

where 'm' and 'n' are integers.

C. Distributive Law:-

Scalar product obeys distributive law

i-e

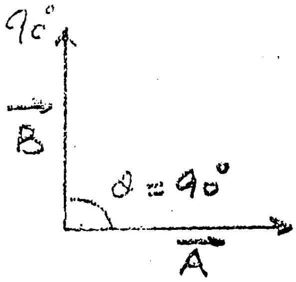
$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$

D. The scalar product of two mutually perpendicular vectors is zero.

i-e $\vec{A} \cdot \vec{B} = AB\cos 90^\circ$

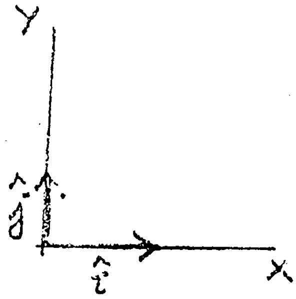
$= AB(0)$

$= Zero.$



In case of unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$.

$$\hat{i}\cdot\hat{j} = |\hat{i}||\hat{j}|\cos 90^\circ$$
$$= 1.1.0$$
$$= \text{Zero}$$



Similarly

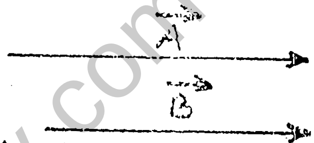
$$\hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = \text{Zero}.$$

(E)- The scalar product of two parallel vectors.

If vectors are parallel to each other, then

$$\theta = 0^\circ$$

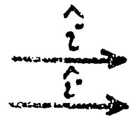
So
$$\vec{A}\cdot\vec{B} = AB\cos 0^\circ$$
$$\vec{A}\cdot\vec{B} = AB$$



In this case scalar product is equal to the product of their magnitudes and scalar product attains its maximum value.

For unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$

$$\hat{i}\cdot\hat{i} = |\hat{i}||\hat{i}|\cos 0$$
$$= 1.1.1$$
$$\hat{i}\cdot\hat{i} = 1$$



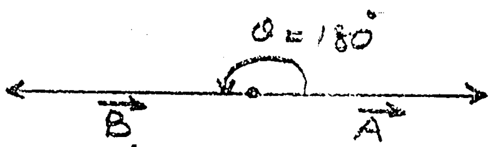
Similarly
$$\hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$

(F)- For antiparallel vectors

In this case

$$\theta = 180^\circ$$

So
$$\vec{A}\cdot\vec{B} = AB\cos 180^\circ$$
$$\vec{A}\cdot\vec{B} = -AB$$

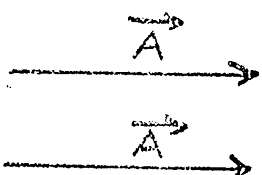


This is the minimum value of scalar prod.

(G)- Scalar Product of two equal vectors

In this case $\theta = 0^\circ$

So
$$\vec{A}\cdot\vec{A} = AA\cos 0^\circ$$
$$\vec{A}\cdot\vec{A} = A^2$$



$\therefore$ The self product of a vector $\vec{A}$ is equal to the square of its magnitude.

(H). Scalar Product interms of Rectangular Comp.

Let $\vec{A}$ and $\vec{B}$ be two vectors which can be written as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

So

$$\vec{A}\cdot\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}).$$

$$= A_x\hat{i}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_y\hat{j}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_z\hat{k}\cdot(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}).$$

$$= A_xB_x(\hat{i}\cdot\hat{i}) + A_xB_y(\hat{i}\cdot\hat{j}) + A_xB_z(\hat{i}\cdot\hat{k}) + A_yB_x(\hat{j}\cdot\hat{i}) + A_yB_y(\hat{j}\cdot\hat{j}) + A_yB_z(\hat{j}\cdot\hat{k}) + A_zB_x(\hat{k}\cdot\hat{i}) + A_zB_y(\hat{k}\cdot\hat{j}) + A_zB_z(\hat{k}\cdot\hat{k})$$

As we know

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$$

and

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = \text{zero}$$

Putting these values in above equation we get

$$\vec{A}\cdot\vec{B} = A_xB_x(1) + A_xB_y(0) + A_xB_z(0) + A_yB_x(0) + A_yB_y(1) + A_yB_z(0) + A_zB_x(0) + A_zB_y(0) + A_zB_z(1).$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

i.e the scalar product of two vectors is equal to the sum of the products of their corresponding components.

NOTE

As $\vec{A}\cdot\vec{B} = AB\cos\theta$

and $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$

By comparing these two equations, angle between two vectors can be found.

$$AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$$

Therefore

$$\boxed{\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{AB}}$$

EXAMPLE 2.4 A force $\vec{F} = 2\hat{i} + 3\hat{j}$ units has its point of application moved from the point $A(1,3)$ to the point $B(5,7)$. Find the work done.

SOLUTION:-

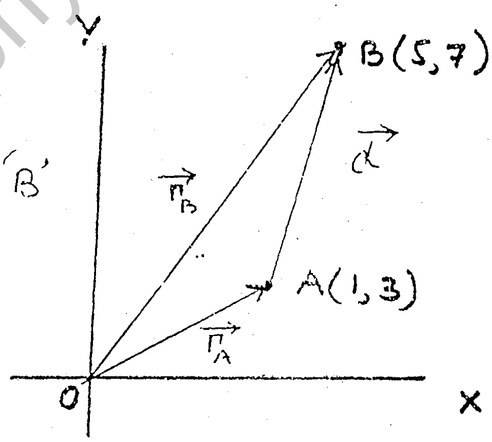
As $\vec{F} = 2\hat{i} + 3\hat{j}$

The position vector of point 'A'

$$\vec{r_A} = \hat{i} + 3\hat{j}$$

The position vector of point 'B'

$$\vec{r_B} = 5\hat{i} + 7\hat{j}$$



From figure

$$\vec{r_B} = \vec{r_A} + \vec{d}$$

$$\therefore \quad \vec{d} = \vec{r_B} - \vec{r_A}$$

$$= (5\hat{i} + 7\hat{j}) - (\hat{i} + 3\hat{j})$$

$$= (5-1)\hat{i} + (7-3)\hat{j}$$

$$\vec{d} = 4\hat{i} + 4\hat{j}$$

$$\text{Work} = \vec{F}\cdot\vec{d} = (2\hat{i} + 3\hat{j})\cdot(4\hat{i} + 4\hat{j})$$

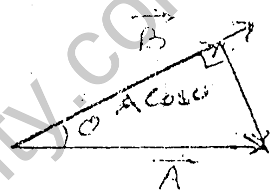
$$= 2\times4(\hat{i}\cdot\hat{i}) + 2\times4(\hat{i}\cdot\hat{j}) + 3\times4(\hat{j}\cdot\hat{i}) + 3\times4(\hat{j}\cdot\hat{j})$$

$\therefore$ Work done $= 8(1) + 8(0) + 12(0) + 12(1)$

Work done $= 8+12 = 20$ units

EXAMPLE 2.5 Find the projection of vector $\vec{A} = 2\hat{i} - 8\hat{j} + \hat{k}$ in the direction of the vector $\vec{B} = 3\hat{i} - 4\hat{j} - 12\hat{k}$.

SOLUTION :- If 'θ' is the angle between $\vec{A}$ and $\vec{B}$, then $A\cos\theta$ is the required projection.



$\therefore$ $\vec{A} \cdot \vec{B} = AB\cos\theta$

$$A\cos\theta = \frac{\vec{A} \cdot \vec{B}}{B}$$

$$A\cos\theta = \vec{A} \cdot \left(\frac{\vec{B}}{B}\right) = \vec{A} \cdot \hat{B} \quad \text{——— (1)}$$

where $\hat{B}$ is the unit vector in the direction of $\vec{B}$.

As $\vec{B} = 3\hat{i} - 4\hat{j} - 12\hat{k}$

then $B = \sqrt{3^2 + (-4)^2 + (-12)^2}$

$B = 13$

Therefore $\hat{B} = \dfrac{\vec{B}}{B} = \dfrac{3\hat{i} - 4\hat{j} - 12\hat{k}}{13}$

So the projection of $\vec{A}$ on $\vec{B}$ = from eq (1)

$$A\cos\theta = \vec{A} \cdot \hat{B}$$

$$= (2\hat{i} - 8\hat{j} + \hat{k}) \cdot \left(\frac{3\hat{i} - 4\hat{j} - 12\hat{k}}{13}\right)$$

$$= \frac{(2)(3) + (-8)(-4) + (1)(-12)}{13}$$

$$= \frac{26}{13} \qquad \left(\because \vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z\right)$$

$\boxed{\text{The projection of } \vec{A} \text{ on } \vec{B} = 2}$ Answer

VECTOR or CROSS PRODUCT

1- **Definition :-** The product of two vectors which results into a vector quantity is called vector product.

2. **Explanation :-**

(a). The vector product is denoted by putting a cross between the vectors. The vector product between vector $\vec{A}$ and vector $\vec{B}$ is written as $\vec{A} \times \vec{B}$. So it is also known as cross product.

(b). **Formula :-** Consider two vectors $\vec{A}$ and $\vec{B}$, let vector $\vec{B}$ be making an angle 'θ' with vector $\vec{A}$ as shown in figure. Then their vector product is equal to another vector $\vec{C}$ which is given as

$$\vec{A} \times \vec{B} = \vec{C} = AB\sin\theta\,\hat{n}$$

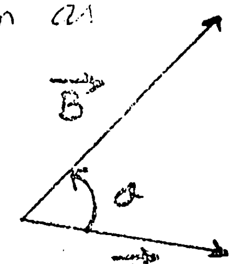


Fig (a)

where $(AB\sin\theta)$ is the magnitude of $\vec{A}\times\vec{B}$ or $\vec{C}$ and '$\hat{n}$' is a unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$. So $\hat{n}$ shows the direction of $\vec{A}\times\vec{B}$ or $\vec{C}$, which can be determined by Right Hand Rule.

(c) **Right Hand Rule :-**

Place the tails of vectors $\vec{A}$ and $\vec{B}$ together, this defines a plane containing $\vec{A}$ and $\vec{B}$. The direction of product vector $\vec{C}$ or $\vec{A}\times\vec{B}$ is perpendicular to this plane. Now

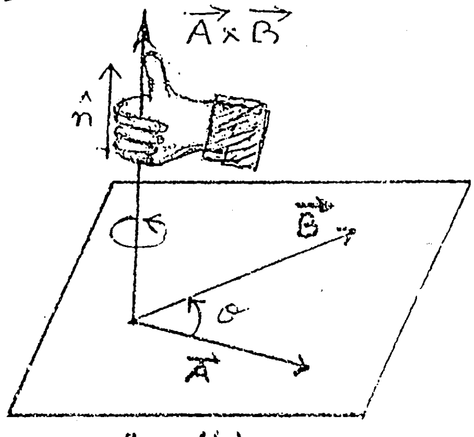


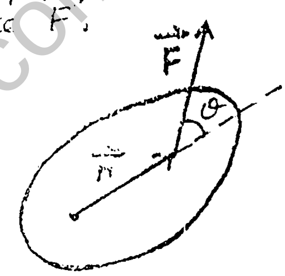
fig (b)

rotate the first vector $\vec{A}$ towards the second vector $\vec{B}$ through the smaller angle ($0 < \theta < \pi$). Curl the fingers of the right hand along the direction of rotation of $\vec{A}$, keeping the thumb erect as shown in figure (b). The direction of the product vector $\vec{A} \times \vec{B}$ or $\vec{C}$ will be along the erect thumb. This rule is called Right Hand Rule.

(d) Examples:-

(i) Torque:- Torque is the cross product of position vector $\vec{r}$ and the force $\vec{F}$.

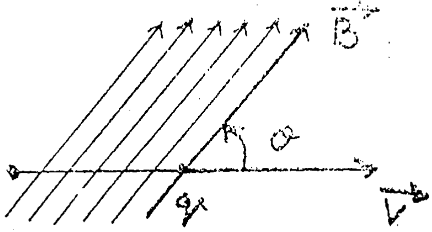
$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$= rF\sin\theta\,\hat{n}$$



(ii) Force on a moving charge in a magnetic field

If a particle of charge 'q' is moving with velocity $\vec{V}$ in a magnetic field of strength $\vec{B}$, then the force $\vec{F}$ acting on it is the cross product of $\vec{V}$ and $\vec{B}$.

$$\vec{F} = q(\vec{V} \times \vec{B})$$
$$\vec{F} = qVB\sin\theta\,\hat{n}$$



(iii). Angular Momentum Angular momentum is the cross product of position vector $\vec{r}$ and the linear momentum $\vec{P}$.

$$\vec{L} = \vec{r} \times \vec{P}$$
$$\vec{L} = rP\sin\theta\,\hat{n}$$

(iv). Relation between Linear and Angular velocity

$$\vec{V} = \vec{\omega} \times \vec{r}$$
$$V = \omega r\sin\theta\,\hat{n}$$

where $\vec{V}$ = linear velocity
$\vec{\omega}$ = Angular velocity
$\vec{r}$ = radius.

CHARACTERISTICS OF CROSS PRODUCT

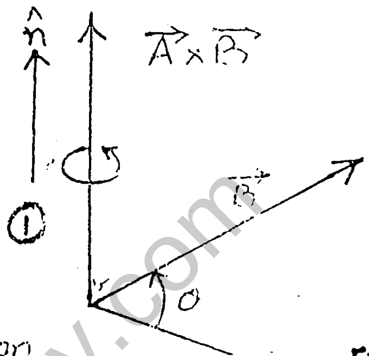
1. Commutative Law:-

The cross product is non commutative i-e it does not obey commative law. $\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$

Proof:-

Consider two vectors $\vec{A}$ and $\vec{B}$. Let $\vec{B}$ be making an angle 'θ' with $\vec{A}$. Then

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n} \quad \text{---} \ (1)$$



Fig(a)

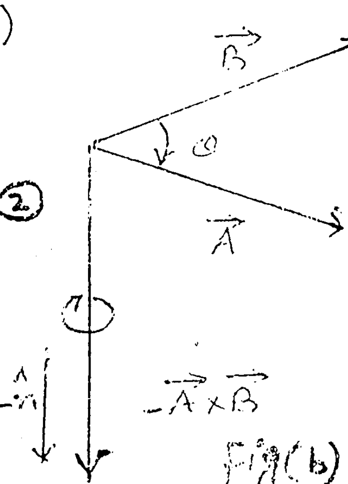
By Right Hand Rule $\vec{A}\times\vec{B}$ is acting in upward direction as shown in figure (a).

Now let $\vec{A}$ be making an angle 'θ' with $\vec{B}$, then

$$\vec{B}\times\vec{A} = BA\sin\theta\,(-\hat{n})$$

or

$$\vec{B}\times\vec{A} = AB\sin\theta\,(-\hat{n})$$
$$= -AB\sin\theta\,\hat{n} \quad \text{---} \ (2)$$



Fig(b).

From eq (1) and (2)

$\therefore \quad \vec{B}\times\vec{A} = -\vec{A}\times\vec{B}$

or

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

$\therefore \quad \vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$

This shows that vector product is non commutative

2. Associative Law:-

Vector product is associative,

i-e $\quad m\vec{A}\times n\vec{B} = mn\,\vec{A}\times\vec{B} = \vec{A}\times mn\vec{B} = n\vec{A}\times m\vec{B}$

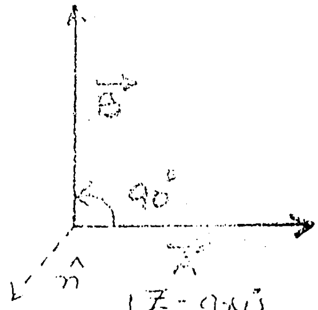
where 'm' and 'n' are numbers.

3. Distributive Law:-

Vector product is distributive over vector addition.

i-e $\quad \vec{A}\times(\vec{B}+\vec{C}+\vec{D}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}+\vec{A}\times\vec{D}$.

4- Cross Product Of Perpendicular Vectors.



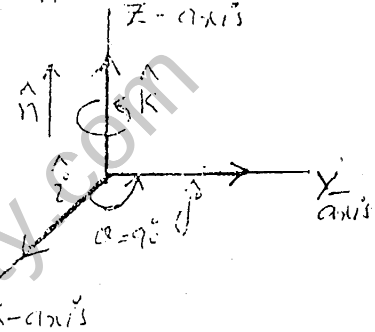
$\theta = 90^\circ$

$\vec{A} \times \vec{B} = AB \sin 90^\circ \hat{n}$

$= AB \hat{n}$

In this case cross product has its maximum value.

For unit vectors.



$\hat{i} \times \hat{j} = |\hat{i}||\hat{j}| \sin 90^\circ \hat{n}$

$= 1 . 1 . 1 \hat{n}$

$\hat{i} \times \hat{j} = \hat{n}$

By Right Hand Rule

$\hat{n} = \hat{k}$

So $\hat{i} \times \hat{j} = \hat{k}$

Similarly

$\hat{j} \times \hat{k} = \hat{i}$ and $\hat{k} \times \hat{i} = \hat{j}$.

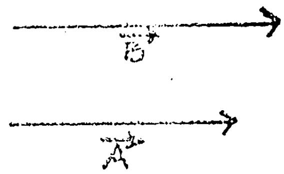
On reversing the orders of the vectors, the directions of product vectors will be reversed

As $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

$\therefore \hat{j} \times \hat{i} = -\hat{k}, \quad \hat{k} \times \hat{j} = -\hat{i}, \quad \hat{i} \times \hat{k} = -\hat{j}$.

5. Cross Product of Parallel Vectors.

The cross product of two parallel vectors is null, because for such vectors $\theta = 0^\circ$ and $\theta = 180^\circ$ (for anti-parallel vectors)



Hence

$\vec{A} \times \vec{B} = AB \sin 0^\circ \hat{n} = AB \sin 180^\circ \hat{n}$

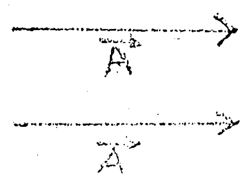
$= AB (0) \hat{n} = \vec{O}$.

6- Cross Product Of Equal Vectors.

In this case $\theta = 0^\circ$

$\vec{A} \times \vec{A} = AA \sin 0^\circ \hat{n}$

$\vec{A} \times \vec{A} = \vec{O}$



For Unit Vectors :-

$$\hat{i}\times\hat{i} = |\hat{i}||\hat{i}|\sin 0^\circ\,\hat{n} = 1.1.0\,\hat{n} = 0$$

Similarly $\hat{j}\times\hat{j} = 0$ and $\hat{k}\times\hat{k} = 0$

7. CROSS PRODUCT INTERMS OF RECTANGULAR COMPONENTS

Two vectors $\vec{A}$ and $\vec{B}$ interms of rectangular components are given as:

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}.$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}.$$

$$\vec{A}\times\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}).$$
$$= A_x\hat{i}\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_y\hat{j}\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}) + A_z\hat{k}\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k}).$$

$$= A_xB_x(\hat{i}\times\hat{i}) + A_xB_y(\hat{i}\times\hat{j}) + A_xB_z(\hat{i}\times\hat{k}) + A_yB_x(\hat{j}\times\hat{i}) + A_yB_y(\hat{j}\times\hat{j}) + A_yB_z(\hat{j}\times\hat{k}) + A_zB_x(\hat{k}\times\hat{i}) + A_zB_y(\hat{k}\times\hat{j}) + A_zB_z(\hat{k}\times\hat{k})$$

As we know that

$$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \text{Zero}$$

and

$$\hat{i}\times\hat{j} = \hat{k} \qquad \hat{j}\times\hat{i} = -\hat{k}$$
$$\hat{j}\times\hat{k} = \hat{i} \qquad \hat{k}\times\hat{j} = -\hat{i}$$
$$\hat{k}\times\hat{i} = \hat{j} \qquad \hat{i}\times\hat{k} = -\hat{j}$$

$\therefore$ we get

$$\vec{A}\times\vec{B} = A_xB_y\hat{k} - A_xB_z\hat{j} - A_yB_x\hat{k} + A_yB_z\hat{i} + A_zB_x\hat{j} - A_zB_y\hat{i}.$$

$$\vec{A}\times\vec{B} = \hat{i}(A_yB_z - A_zB_y) + \hat{j}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)$$

This expansion can also be written in determinant form as

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

8. Magnitude of Vector Product is equal to the area of Parallelogram.

As

$\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$

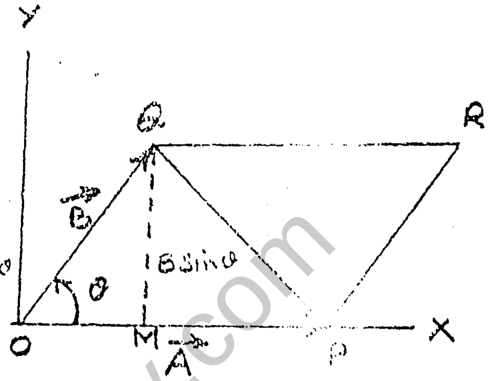
$|\vec{A} \times \vec{B}| = AB\sin\theta$

$= A(B\sin\theta)$

From figure

$OP = A$ and

$OQ = B$ then $QM = B\sin\theta$



$|\vec{A} \times \vec{B}| = (OP)(QM)$

$=$ (Base of $\triangle OPQ$)(Height)

$= 2 \times \frac{1}{2}$ (Base of $\triangle OPQ$)(Height of $\triangle OPQ$)

$= 2\left\{\frac{1}{2}\right.$ (Base of $\triangle OPQ$)(Height of $\triangle OPQ$)$\left.\right\}$

$= 2$ (Area of $\triangle OPQ$)

($\because$ Area of Triangle $= \frac{1}{2}$(base)(height))

$|\vec{A} \times \vec{B}| =$ Area of $\triangle OPQ$ + Area of $\triangle OPQ$

As $\triangle OPQ \cong \triangle PQR$

So

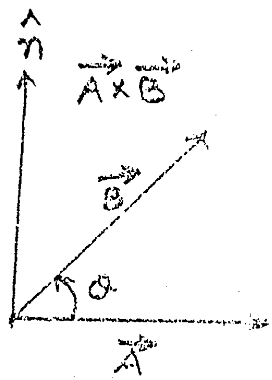
$|\vec{A} \times \vec{B}| =$ Area of $\triangle OPQ$ + Area of $\triangle PQR$

$|\vec{A} \times \vec{B}| =$ Area of Parallelogram OPQR

NOTE :-

Outer Product

The vector or Cross product is also known as Outer product because in this case resultant of $\vec{A} \times \vec{B}$ is lying out of the plane containg $\vec{A}$ and $\vec{B}$.



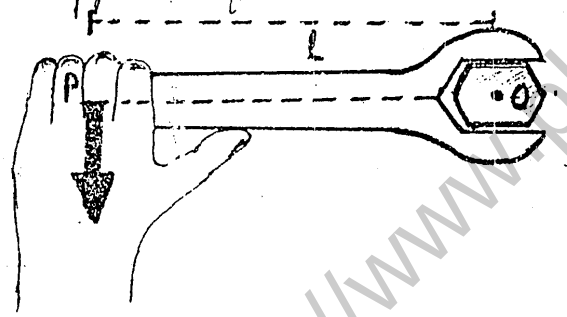
i.e.

$\vec{A} \times \vec{B} = AB\sin\theta\,\hat{n}$.

By Right Hand Rule '$\hat{n}$' indicates the direction of $\vec{A} \times \vec{B}$ acting out of the plane.

TORQUE

1. **Introduction :-** In order to produce the rotation in a body about a certain axis, force should be applied in such a way that the line of action does not pass through the axis of rotation. Such a force produces turning effect in the body. For example turning effect is produced when a nut is tightened with a spanner as shown in fig (a). This turning effect increases when you push harder on the spanner. It also depends on the length of the spanner, the longer the handle of the spanner the greater will be the turning effect of an applied force.



The nut is easy to turn with a spanner.

2. **Definition :-** Torque is defined as " The turning effect of a force produced in a body about a certain point is called Torque or moment of force.

3. **Explanation :-**

(a) - Torque acting on a body can be calculated by multiplying force and the perpendicular distance from its line of action to the pivot which is the point 'O' (as shown in above fig.) around which the body (spanner) rotates. This distance OP is called moment arm 'l'

Moment Arm :- The perpendicular distance of line of action of a force from axis of rotation (pivot) is called Moment Arm.

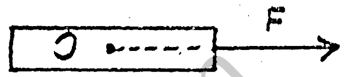
Pivot Point:: A point around which a body can rotate.

Thus magnitude of torque represented by 'τ' is

$$\tau = (\text{Force})(\text{Moment Arm})$$

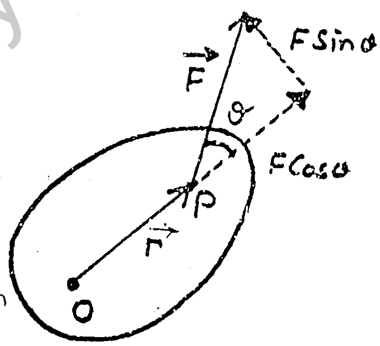
$$\boxed{\tau = F\,l} \qquad (1)$$

When the line of action of the applied force passes through the pivot point, then perpendicular distance of force from pivot pt. is zero i-e moment arm $l = 0$, so in this case torque is zero ($\because \tau = Fl = F(0) = \text{Zero}$)



(b)- TORQUE ACTING ON A RIGID BODY

Consider the torque due to a force $\vec{F}$ acting on a rigid body. Let the force $\vec{F}$ acts on rigid body at point 'P' whose position vector relative to pivot 'O' is $\vec{r}$. The force $\vec{F}$ can be resolved into two rectangular components, $F\sin\theta$ perpendicular to $\vec{r}$ and $F\cos\theta$ along the direction of $\vec{r}$.

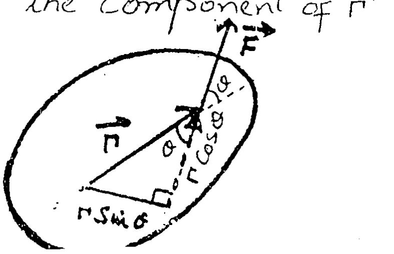


The torque due to $F\cos\theta$ about pivot 'O' is zero as its line of action passes through point 'O'. Therefore the torque due to $\vec{F}$ is equal to the torque due to $F\sin\theta$ only about 'O'.

As
$$\text{Torque} = (\text{Force})(\text{Moment Arm})$$

$$\tau = F\sin\theta\, r = rF\sin\theta \qquad (2)$$

Similarly if we resolve the position vector $\vec{r}$ into its components, then '$r\cos\theta$' is the component of $\vec{r}$ in the direction of $\vec{F}$. Since both are in same direction therefore torque due to '$r\cos\theta$' will be zero.



So in this case $r\sin\theta$ is the component of $\vec{r}$ perpendicular to $\vec{F}$. i-e moment arm $l = r\sin\theta$

$\therefore$

$$\tau = (F)(r\sin\theta) = rF\sin\theta \quad \text{——— (3)}$$

where 'θ' is the angle between $\vec{r}$ and $\vec{F}$.

From equation (2) and (3)

$$\tau = rF\sin\theta$$

Multiplying on both sides by a unit vector '$\hat{n}$'

$$\tau\hat{n} = rF\sin\theta\,\hat{n}$$

$$\therefore \quad \boxed{\vec{\tau} = \vec{r}\times\vec{F}} \quad \text{——— (4)}$$

(c) - SECOND DEFINITION :-

Torque can also be defined as, the vector product of position vector $\vec{r}$ and the force $\vec{F}$ is called torque.

As $\vec{\tau} = \vec{r}\times\vec{F} = rF\sin\theta\,\hat{n}$

$(rF\sin\theta)$ is the magnitude of the torque. The direction of the torque represented by '$\hat{n}$' is perpendicular to the plane containing $\vec{r}$ and $\vec{F}$ given by right hand rule.

(d) UNITS :-

S.I. unit — Its S.I unit is Nm

C.G.S. unit — Its CGS unit is dyne cm

F.P.S. unit — Its FPS unit is pound foot

(e) DIMENSIONS :-

Dimensions of torque is $[ML^2T^{-2}]$

(f) TORQUE IS THE COUNTER PART OF FORCE

Torque is the counter part of force for rotational motion. Because

$F = ma$, from this equation force determines the linear acceleration produced in a body.

In rotational motion the equivalent equation is $\tau = I\alpha$, which shows that torque acting on a body determines its angular acceleration.

If a body is at rest or moving with constant angular velocity, then angular acceleration will be zero (i.e $\alpha = 0$). In this case the torque acting on the body will be zero ($\tau = I\alpha = I(0) = 0$).

(g). THIRD DEFINITION:-

Torque may also be defined as a physical quantity which produces an angular acceleration in a body about pivot point (axis of rotation) is called torque.

(h). SIGN CONVENTIONS

(i). **Positive Torque:-** The torque which rotates the body in anticlockwise direction is considered as positive torque.

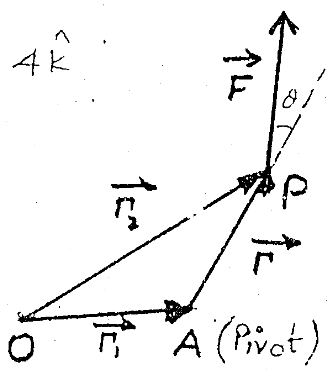
(ii). **Negative Torque:-** The torque which rotates the body in clockwise direction is considered as negative torque.

EXAMPLE 2.6 The line of action of a force $\vec{F}$ acts through a point 'P' of a body whose position vector is $\hat{i} - 2\hat{j} + \hat{k}$. If $\vec{F} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ (N), determine the torque about the point 'A' whose position vector (in meters) is $2\hat{i} + \hat{j} + \hat{k}$

SOLUTION:- $\vec{F} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

The position vector of point $A = \vec{r}_1 = 2\hat{i} + \hat{j} + \hat{k}$

The position vector of point $P = \vec{r}_2 = \hat{i} - 2\hat{j} + \hat{k}$



The position vector of P relative to $A = \vec{AP} = \vec{r} = \vec{r}_2 - \vec{r}_1$

$$\vec{r} = (\hat{i} - 2\hat{j} + \hat{k}) - (2\hat{i} + \hat{j} + \hat{k})$$

$$= (\hat{i} - 2\hat{i}) + (-2\hat{j} - \hat{j}) + (\hat{k} - \hat{k})$$

$$\vec{r} = -\hat{i} - 3\hat{j}$$

The torque about pivot 'A' $= \vec{\tau}_A = \vec{r} \times \vec{F}$

$$\vec{\tau}_A = (-\hat{i} - 3\hat{j}) \times (2\hat{i} - 3\hat{j} + 4\hat{k})$$

$$= (-1)(2)(\hat{i}\times\hat{i}) + (-1)(-3)(\hat{i}\times\hat{j}) + (-1)(4)(\hat{i}\times\hat{k}) + (-3)(2)(\hat{j}\times\hat{i}) + (-3)(-3)(\hat{j}\times\hat{j}) + (-3)(4)(\hat{j}\times\hat{k})$$

$$= -2(0) + 3(\hat{k}) - 4(-\hat{j}) - 6(-\hat{k}) + 9(0) - 12(\hat{i})$$

$$= -12\hat{i} + 4\hat{j} + 3\hat{k} + 6\hat{k}$$

$$\boxed{\vec{\tau}_A = -12\hat{i} + 4\hat{j} + 9\hat{k}}$$

EQUILIBRIUM OF FORCES

- **EQUILIBRIUM :-** If a body is at rest or moves with uniform velocity, then the body is said to be in equilibrium.

  Types of Equilibrium :- There are two types.

  1- **STATIC EQUILIBRIUM :-** A body at rest is said to be in static equilibrium.
     e.g - a book lying on a table
     - a body attached with a string at rest.

  2- **DYNAMIC EQUILIBRIUM :-** When a body is moving with uniform velocity, then the body is in the state of dynamic equilibrium.
     e.g - a car moving with uniform velocity
     - jumping of paratrooper.

- **CONDITIONS OF EQUILIBRIUM :-**

  1- There are two conditions.

  1- **FIRST CONDITION OF EQUILIBRIUM**

  (a)- **Introduction :-** A body at rest or moving with uniform velocity has zero acceleration. From Newton's law of motion $(\vec{F} = m\vec{a})$ the resultant force will also be zero. $(F = 0)$

(b). Statement :- The vector sum of all the forces acting on a body must be equal to zero. This is known as first condition of equilibrium

(C). Explanation :-

(i) Mathematically :-

Suppose there are 'n' number of forces acting on a body, then the body will be in equilibrium, when

$$\vec{F_1} + \vec{F_2} + \vec{F_3} + \cdots + \vec{F_n} = 0$$

or Using the mathematical symbol $\Sigma \vec{F}$ for sum of all the forces

$$\therefore \quad \Sigma \vec{F} = 0$$

(ii). For Coplanar Forces :-

This condition is expressed usually in terms of x and y components of the forces.
As we know that x-component of the resultant force $\vec{F}$ equals to the sum of x-components of all the forces acting on the body

$$F_{1x} + F_{2x} + F_{3x} + \cdots + F_{nx} = 0$$

or

$$\Sigma F_x = 0$$

Similarly the y-component of the resultant force $\vec{F}$ equals to the sum of y-components of all the forces acting on the body.

$$F_{1y} + F_{2y} + F_{3y} + \cdots + F_{ny} = 0$$

or

$$\Sigma F_y = 0$$

(iii). Sign Conventions :-

It may be noted that if the rightward forces are taken as positive then leftward forces are taken as negative. Similarly if upward forces are taken as positive then downward forces are taken as negative.

EXAMPLE 2.7 A load is suspended by two cords as shown in figure. Determine the maximum load that can be suspended at 'P', if maximum breaking stress of the chord used is 50 N.

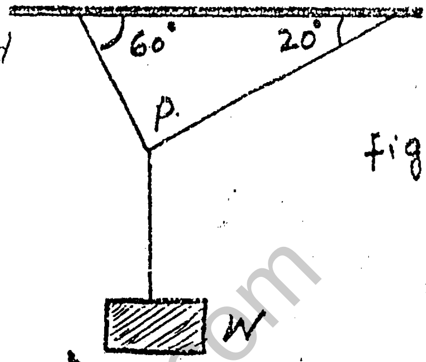


fig (a)

SOLUTION:- For using conditions of equilibrium, all the forces acting at point 'P' are shown in figure (b), where 'W' is assumed to be the maximum weight which can be suspended. The inclind forces can now be easily resolved along x and y-directions.

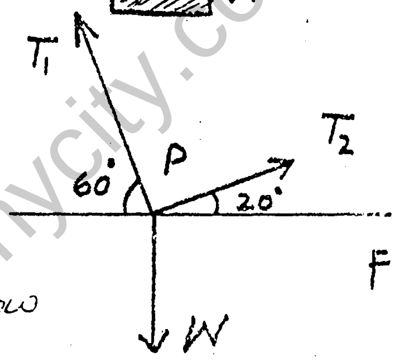
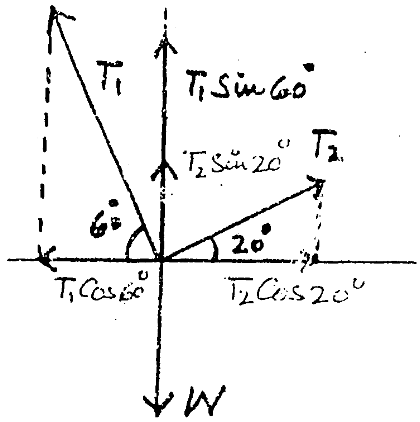


Fig (b)



Applying

$$\Sigma F_x = 0$$

$$T_2 \cos 20^\circ - T_1 \cos 60^\circ = 0$$

$$T_1 \cos 60^\circ = T_2 \cos 20^\circ$$

$$T_1 = \frac{T_2 \cos 20^\circ}{\cos 60^\circ}$$

$$T_1 = 1.88\, T_2$$

$$\Rightarrow \quad T_1 > T_2$$

$\therefore$ T_1 has the maximum stress.

As maximum stress is given 50N

so $T_1 = 50N$, then

$$T_1 = 1.88 T_2 \quad \Rightarrow \quad T_2 = \frac{50}{1.88} = 26.6N$$

Now applying

$$\Sigma F_y = 0$$

$$T_1 \sin 60^\circ + T_2 \sin 20^\circ - W = 0$$

Putting the values

$$50N \times 0.866 + 26.6N \times 0.34 = W$$

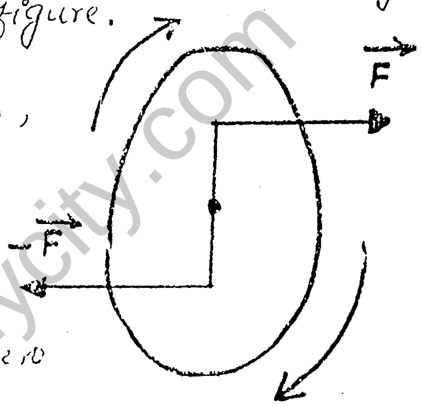
$$\boxed{W = 52N}$$

EQUILIBRIUM OF TORQUES

SECOND CONDITION OF EQUILIBRIUM :-

(a). Introduction :-

Let two equal and opposite forces are acting on a rigid body as shown in figure. Although the first condition of equilibrium is satisfied $\Sigma \vec{F} = 0$, yet it may rotate having clockwise turning effect. For angular acceleration (α) to be zero, the net torque acting on the body should be zero i-e $\tau = I\alpha = 0$



(b). Statement :-

For a body in equilibrium, the vector sum of all the Torques acting on a body about any arbitrary axis is zero. This is known as second condition of equilibrium

(c). Explanation :-

(i). Mathematically :-

$$\boxed{\Sigma \vec{\tau} = 0}$$

(ii). Sign Convention :-

The counter clockwise torques are taken as positive and clockwise torques as negative. The position of the axis is quite arbitrary.

TRANSLATIONAL EQUILIBRIUM :-

When the first condition of equilibrium is satisfied i-e $\Sigma \vec{F} = 0$, then there is no linear acceleration ($a = 0$), the body will be in Translational Equilibrium.

ROTATIONAL EQUILIBRIUM:-

When second condition ($\Sigma\tau = 0$) is satisfied, there is no angular acceleration and body will be in rotational equilibrium.

COMPLETE EQUILIBRIUM:-

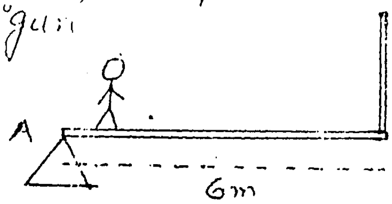
For a body to be in complete equilibrium both conditions should be satisfied, i-e both linear and angular accelerations should be zero.

i-e

1- $\Sigma\vec{F} = 0 \quad \Sigma F_x = 0$ and $\Sigma F_y = 0$

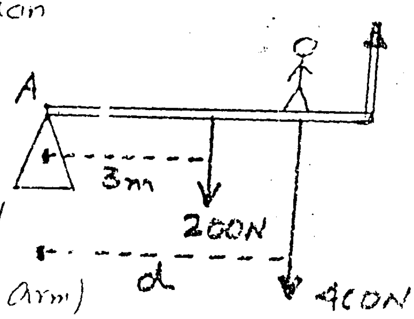
2- $\Sigma\vec{\tau} = 0$

EXAMPLE 2.8 A uniform beam of 200N is supported horizontally as shown. If the breaking stress of the rope is 400N, how far can the man of weight 400N walk from point 'A' on the beam as shown in figure



SOLUTION:- Let breaking point is at a distance 'd' from the pivot point 'A'. The force diagram of the situation is illustrated in figure(b).

By applying 2nd condition of equilibrium about point 'A'.



$\Sigma\tau = 0$

$(\tau)_{upward} = (\tau)_{downward}$

where $\tau = (\text{Force})(\text{Moment arm})$

$\therefore \quad 400N \times 6m = 400N \times d + 200N \times 3m$

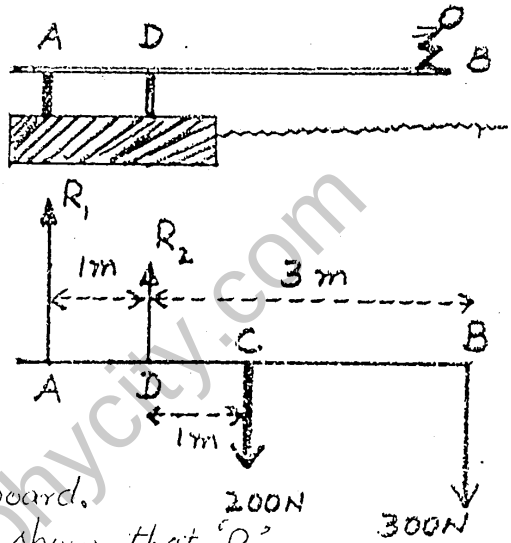
$\Rightarrow \quad 400N \times d = 2400Nm - 600Nm$

$$\boxed{d = 4.5m}$$

EXAMPLE 2.9 A boy weighing 300N is standing at the edge of a uniform diving board 4m in length. The weight of the board is 200N. Find the forces exerted by pedestals on the board.

SOLUTION:-

The forces which act on the diving board are shown in the figure below. The weight 200N of the uniform diving board is acting at pt. C (the centre of gravity). R_1 and R_2 are the reaction forces exerted by the pedestals on the board.



A little consideration will show that 'R_1' is in the wrong direction, because the board must be actually pressed down in order to keep it in equilibrium. We shall see that this mistake will be automatically corrected by calculation.

Applying conditions of equilibrium.

$\Sigma F_x = 0$ (x-components of forces are zero)

$\Sigma F_y = 0$

$R_1 + R_2 - 200 - 300 = 0$

$R_1 + R_2 = 500N$ ——— (1)

$\Sigma \tau = 0$ (pivot at pt. D) $\tau = Fl$

$-R_1 \times AD - R_2(0) - 200 \times DC - 300 \times BD = 0$

$-R_1(1m) - 0 - 200N(1m) - 300N(3m) = 0$

$R_1 = -1100N$

Putting it in eq (1)

$-1100N + R_2 = 500N$

$\boxed{R_2 = 1600N}$

The -ve sign of R_1 shows that it is directed downward in order to keep the body in equilibrium.

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The End