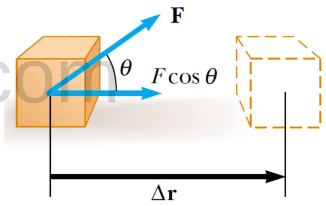
WORK AND ENERGY

Q # 1. Define the term work. Also derive the expression of work done by a variable force.

The work done by a constant force is defined as:

The product of magnitude of displacement and
the component of the force in the direction of the displacement.

Consider an object which is being pulled by a constant force **F** at an angle θ to the direction of motion. The force displaces the object from one point to another point through displacement $\Delta\mathbf{r}$. The work done W by the force is described as:



$$W = (F\cos\theta)d = Fd\cos\theta$$

$$W = \mathbf{F}.\mathbf{d}$$

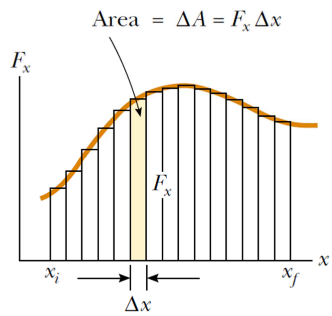
Thus the work done by a force is the dot product of force and displacement. It is a scalar quantity and its IS unit is joule (J = N m).

- If $\theta < 90^\circ$, work is said to be positive
- If $\theta = 90^\circ$, no work is done
- If $\theta > 90^\circ$, the work done is said to be negative

Work Done by a Variable Force

Consider a variable force act on a body which displaces it from one point to another point.

We want to find out the expression of work done by the variable force. For this we divide the path followed by the object into n short intervals of displacements $\Delta\mathbf{x_1}, \Delta\mathbf{x_2}, \Delta\mathbf{x_3}, \ldots\ldots\ldots, \Delta\mathbf{x_n}$ and $\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \ldots\ldots\ldots$ $\mathbf{F_n}$ are the forces acting during these intervals. If $\Delta W_1, \Delta W_2, \Delta W_3, \ldots\ldots\ldots, \Delta W_n$ are the work done during displacement interval $\Delta\mathbf{x_1}, \Delta\mathbf{x_2}, \Delta\mathbf{x_3}, \ldots\ldots\ldots, \Delta\mathbf{x_n}$ respectively, then the total work done W_{total} by the variable force will be:



$$W_{total} = \Delta W_1 + \Delta W_2 + \Delta W_3 + \ \ldots\ldots\ldots + \Delta W_n \qquad \text{------------- } \mathbf{(1)}$$

Now

The work done during displacement interval $\Delta\mathbf{x_1} = \Delta W_1 = \mathbf{F_1}.\Delta\mathbf{x_1} = F_1\,\Delta x_1\cos\theta_1$

The work done during displacement interval $\Delta\mathbf{x_2} = \Delta W_2 = \mathbf{F_2}.\Delta\mathbf{x_2} = F_2\,\Delta x_2\cos\theta_2$

The work done during displacement interval $\Delta\mathbf{x_3} = \Delta W_3 = \mathbf{F_3}.\Delta\mathbf{x_3} = F_3\,\Delta x_3\cos\theta_3$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

The work done during displacement interval $\Delta\mathbf{x_n} = \Delta W_n = \mathbf{F_n}.\Delta\mathbf{x_n} = F_n\,\Delta x_n\cos\theta_n$

Putting value in equation (1), we get:

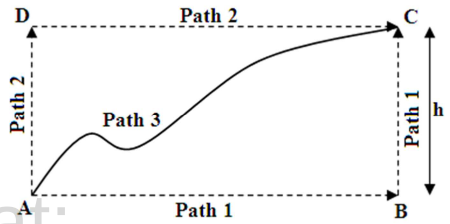
$$W_{total} = \Delta W_1 + \Delta W_2 + \Delta W_3 + \ \ldots\ldots\ldots + \Delta W_n$$

$$W_{total} = F_1\,\Delta x_1\cos\theta_1 + F_2\,\Delta x_2\cos\theta_2 + F_3\,\Delta x_3\cos\theta_3 + \ \ldots\ldots\ldots + F_n\,\Delta x_n\cos\theta_n$$

$$W_{total} = \sum_{i=1}^{n} F_i\,\Delta x_i\cos\theta_i$$

This is the expression of work done by a variable force.

Q # 2. Show that the work done in earth's gravitational field is independent of path followed.



Consider an object of mass m being displaced with constant velocity from point A to point C along various paths in the presence of a gravitational force. In this case the gravitational force is equal to the equal to weight mg of the object.

Work done along path 1

The work done W_{ABC} by the gravitational force along the path ABC can be split into two parts i.e., W_{AB} and W_{BC}.

$W_{ABC} = W_{AB} + W_{BC}$ ------------- **(1)**

The work W_{AB} is zero because weight $m\mathbf{g}$ is perpendicular to this path. The work $W_{BC} = -m\mathbf{g}h$; the negative sign is due to the fact that $m\mathbf{g}$ is opposite to that of displacement. Thus the equation (1) will become:

$W_{ABC} = -m\mathbf{g}h$ ------------- **(2)**

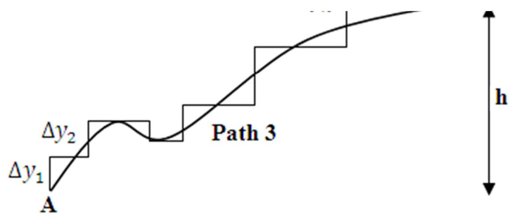
Work done along path 2

If we consider the path ADC, the work done along path AD is $W_{AD} = -m\mathbf{g}h$. Since the work done along path DC is $W_{DC} = 0$, therefore:

$W_{ADC} = W_{AD} + W_{DC} = -m\mathbf{g}h + 0 = -m\mathbf{g}h$ ------------- **(3)**

Work done along path 3

To find out the work done along this curved path, we divide it into series of horizontal and vertical steps as shown in the figure. There is no work done along horizontal steps, because $m\mathbf{g}$ is perpendicular to the displacement for these steps. The work done by the force of gravity along vertical displacements will be:



$W_{AC} = -mg(\Delta y_1 + \Delta y_2 + \dots\dots\dots + \Delta y_n)$

As $\Delta y_1 + \Delta y_2 + \dots\dots\dots + \Delta y_n = h$

Therefore

$W_{AC} = -mgh$ ------------- **(4)**

Thus from equation (2), (3) and (4), it is proved that work done in the earth's gravitational field is independent of the path followed.

Q # 3. Show that work done in a close path is zero or Show that earth's gravitational field is conservative field.

Consider an object of mass m being displaced with constant velocity along the path ABCDA in the presence of a gravitational force. In this case the gravitational force is equal to the equal to weight mg of the object.

The work done along this close path ABCDA will be:

$W_{ABCDA} = W_{AB} + W_{BC} + W_{CD} + W_{DA}$ ------------- **(1)**

- The work W_{AB} and W_{CD} is zero because weight $m\mathbf{g}$ is perpendicular to displacements along these paths.
- The work $W_{BC} = -m\mathbf{g}h$; the negative sign is due to the fact that $m\mathbf{g}$ is opposite to that of displacement.
- The work done along path DA is $W_{DA} = m\mathbf{g}h$

Putting value in equation (1), we get:

$$W_{ABCDA} = W_{AB} + W_{BC} + W_{CD} + W_{DA}$$
$$W_{ABCDA} = 0 - m\mathbf{g}h + 0 + m\mathbf{g}h$$
$$W_{ABCDA} = 0$$

So the work done along close path is zero. Hence the earth's gravitational field is conservative field.

Q # 4. Define the term power. Show that power is the dot product of force and velocity.

Power

The rate at which the work is being done is called power. It is a scalar quantity and its SI unit is watt (W).

If the work ΔW is done in time interval Δt, then the average power P_{av} during this time interval is described as:

$$P_{av} = \frac{\Delta W}{\Delta t}$$

If the work is expressed as function of time, then the instantaneous power P at any instant is defined as:

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t}$$

Relationship between Power and Velocity

If the work ΔW is done in time interval Δt, then the instantaneous power P at any instant is defined as:

$$P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t}$$

Since work done $\Delta W = \mathbf{F}.\Delta\mathbf{d}$

$$P = \lim_{\Delta t \to 0} \frac{\mathbf{F}.\Delta\mathbf{d}}{\Delta t}$$

As $\lim_{\Delta t \to 0} \frac{\Delta \mathbf{d}}{\Delta t} = \mathbf{v}$

$$P = \lim_{\Delta t \to 0} \mathbf{F}.\mathbf{v}$$

Hence proved that the power is the dot product of force and velocity.

Q # 5. Show that 1 kWh = 3.6 MJ

In electrical measurements, the unit of work is expressed as watt second. However the commercial unit of electrical energy is kilowatt-hour.

$$1\text{ kWh} = 1000\text{ W} \times 3600\text{ s}$$
$$= 3600000\text{ J}$$
$$= 3.6 \times 10^6\text{ J}$$
$$1\text{ kWh} = 3.6\text{ MJ}$$

Hence proved.

Q # 6. Define the term energy. Describe the different types mechanical energy.

Energy

Energy of the body is its capacity to do work. It is a scalar quantity and its SI unit is joule (J). the kinetic energy, gravitational potential energy and elastic potential energy are its different types.

Kinetic Energy

The kinetic energy $(K.E)$ is possessed by a body due to its motion and is given by the formula:

$$K.E = \frac{1}{2}m\mathbf{v}^2$$

Where m is the mass of the body moving with velocity $\mathbf{v}$.

Gravitational Potential Energy

The gravitational potential energy ($P.E$) is possessed by a body due to its position in gravitational field. The potential energy due to gravitational field near the surface of the earth at height h is given by the formula:

$$P.E = mgh$$

Elastic Potential Energy

The elastic potential energy is the energy stored in the spring due to its compressed or stretched state.

Q # 7. State the work energy principle. Also derive its mathematical form.

Statement

Work done on the body equals the change in its kinetic energy.

Consider a body of mass m is moving with velocity v_i. A force F acting through a distance d increases the velocity to v_f, then from the 3rd equation of motion:

$$2ad = \mathrm{v}_f^2 - \mathrm{v}_i^2$$

$$\Rightarrow d = \frac{1}{2a}\left(\mathrm{v}_f^2 - \mathrm{v}_i^2\right) \qquad \text{------------------- } \mathbf{(1)}$$

From the second law of motion:

$$F = ma \qquad \text{------------------- } \mathbf{(2)}$$

Multiplying equation (1) and (2), we have:

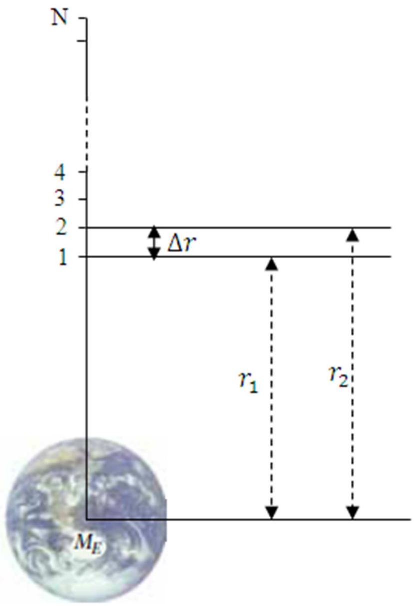
$$Fd = \frac{1}{2}m\left(\mathrm{v}_f^2 - \mathrm{v}_i^2\right)$$

$$Fd = \frac{1}{2}m\mathrm{v}_f^2 - \frac{1}{2}m\mathrm{v}_i^2 \qquad \text{------------------- } \mathbf{(3)}$$

Where the left hand side of the above equation gives the work done on the body and the right hand side gives the change in kinetic energy of the body. The equation (3) is the mathematical form of work energy principle.

Q # 8. What do you know about absolute potential energy? Also derive its expression.

Absolute Potential Energy



The absolute potential energy of an object at a certain point is the work done by the gravitational force in displacing the object from that position to infinity (where the force of gravity becomes zero).

The relation for the calculation for the potential energy $P.E = mgh$, is true only near the surface of the earth where the gravitational force is nearly constant. But if the object is displaced through the large distances, then the gravitational force will not remains constant, since it varies inversely to the square of the distance.

In order to overcome this difficulty, we divide the whole distance into small steps each of length Δr so that the value of the force remains constant for each small step. Hence the total work done can be calculated by adding all the work done during all these steps.

The gravitational force at the center of this step is

$$F = G\frac{Mm}{r^2} \quad \text{--------------------} \quad \mathbf{(1)}$$

Where G is the gravitational constant, m is the mass of the object and M is the mass of the earth.

If r_1 and r_2 are the distances of points 1 and 2 respectively, from the center O of earth. The distance between the center of 1st step and center of the earth will be:

$$r = \frac{r_1 + r_2}{2}$$

If $\Delta r = r_2 - r_1 \Rightarrow r_2 = r_1 + \Delta r$

Hence

$$r = \frac{r_1 + r_1 + \Delta r}{2} = \frac{2r_1 + \Delta r}{2} = r_1 + \frac{\Delta r}{2}$$

Squaring above equation we get:

$$r^2 = \left(r_1 + \frac{\Delta r}{2}\right)^2 = {r_1}^2 + 2r_1\frac{\Delta r}{2} + \left(\frac{\Delta r}{2}\right)^2$$

As $\Delta r^2 \ll {r_1}^2$, so we can neglect the term $\left(\frac{\Delta r}{2}\right)^2$

Hence $r^2 = {r_1}^2 + r_1\Delta r = {r_1}^2 + r_1(r_2 - r_1) = r_1 r_2$

Thus the equation (1) will become:

$$F = G\frac{Mm}{r_1 r_2}$$

As the force is assumed to be constant during the interval Δr, so the work done in displacing the object from point 1 to point 2 will be:

$$W_{1\to2} = \mathbf{F}.\mathbf{\Delta r} = F\Delta r\cos 180^\circ = -F\Delta r = -G\frac{Mm}{r_1 r_2}(\Delta r) = -G\frac{Mm}{r_1 r_2}(r_2 - r_1)$$

$$W_{1\to2} = -GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Similarly the work done during the second step in which the body is displaced from point 2 to point 3 is

$$W_{2\to3} = -GMm\left(\frac{1}{r_2} - \frac{1}{r_3}\right)$$

The work done during the last step is

$$W_{(N-1)\to N} = -GMm\left(\frac{1}{r_{N-1}} - \frac{1}{r_N}\right)$$

Hence, the total work done in displacing the body form point 1 to point N is calculated bt adding up the work dome during all these steps.

$$W_{total} = W_{1\to2} + W_{2\to3} + \dots\dots\dots\dots + W_{(N-1)\to N}$$

$$W_{total} = -GMm\left[\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \left(\frac{1}{r_2} - \frac{1}{r_3}\right) + \dots\dots\dots\dots + \left(\frac{1}{r_{N-1}} - \frac{1}{r_N}\right)\right]$$

On simplification, we get:

$$W_{total} = -GMm\left(\frac{1}{r_1} - \frac{1}{r_N}\right)$$

If the point N is situated at infinite distance from the earth, so $r_N = \infty$

$$W_{total} = -GMm\left(\frac{1}{r_1} - \frac{1}{\infty}\right) = -GMm\left(\frac{1}{r_1} - 0\right)$$

$$W_{total} = -\frac{GMm}{r_1}$$

Therefore the general expression for the gravitational potential energy of a body situated at distance r form the center of the earth is

$$U = -\frac{GMm}{r}$$

The negative sign shows that the earth gravitation field for mass m is attractive.

Q # 9. Define the term escape velocity. Also derive its expression.

Escape Velocity

The velocity given to the object on the surface of the earth so that it escapes the earth gravitational field and reaches at an infinite distance from the surface of the earth is called the escape velocity.

Derivation of Mathematical Expression for Escape Velocity

Consider a object of mass m which is given the initial velocity v_{esc} to escape the earth's gravitational field. The K.E corresponding to the initial velocity is will be

$$Initial\ K.E = \frac{1}{2} m\, v_{esc}^2$$

We know that the work done in lifting a body from the earth surface to infinite distance is equal to increase in its potential energy.

$$Increase\ in\ P.E\ = 0 - \left(-G\frac{Mm}{R^2}\right) = G\frac{Mm}{R}$$

Where M and R are the mass and radius of the earth,

The body will escape out of gravitational field if the initial K.E. of the body is equal to the increase in P.E. of the body in lifting it up to infinity. Then

$$\frac{1}{2} m\, v_{esc}^2 = G\frac{Mm}{R}$$

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

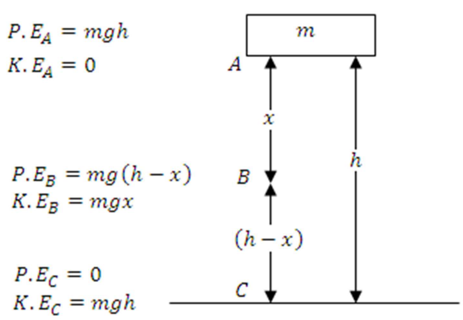
As $g = \frac{GM}{R^2}$

Hence $v_{esc} = \sqrt{2gR}$

The value of v_{esc} comes out to be approximately 11 kms^{-1}.

Q # 10. Describe the inter-conversion of potential energy into kinetic energy for a free falling object.

Consider a body of mass m a rest, at a height h above the surface of earth as shown in the figure. We want to discuss the inter-conversion of potential energy into kinetic energy for a free falling object under the action of gravity.



Energy of the body at point A

The potential energy of the body at point A is $P.E_A = mgh$ and the kinetic energy at this point is $K.E_A = 0$. Thus the total energy of the object at this position will be:

$$T.E_A = P.E_A + K.E_A$$

$$T.E_A = mgh + 0 = mgh$$

Energy of the body at point B

During downward motion, the object passes through point B. When the object is at point B, the body has fallen through a distance x.

$$P.E_B = mg(h - x)$$

And $K.E_B = \frac{1}{2}mv_B^2$

The velocity at point B can be calculated using 3rd equation of motion.

$$v_f^2 - v_i^2 = 2aS$$

Here $v_f = v_B, v_i = 0, S = x, a = g$

$$v_B^2 - 0 = 2gx$$

$$v_B = \sqrt{2gx}$$

$$K.E_B = \frac{1}{2}m(2gx) = mgx$$

The total energy at point is

$$T.E_B = P.E_B + K.E_B$$

$$T.E_B = mg(h - x) + mgx = mgh$$

Energy of the body at point C

At point C, just before the body strikes the earth, the potential energy and the kinetic energy will be:

$$P.E_C = 0$$

$$K.E_C = \frac{1}{2}mv_C^2$$

The velocity at point C can be calculated using 3rd equation of motion.

$$v_f^2 - v_i^2 = 2aS$$

Here $v_f = v_C, v_i = 0, S = h, a = g$

$$v_C^2 - 0 = 2gh$$

$$v_C = \sqrt{2gh}$$

$$K.E_C = \frac{1}{2}m(2gh) = mgh$$

The total energy at point is

$$T.E_C = P.E_C + K.E_C$$

$$T.E_C = 0 + mgh = mgh$$

Conclusion

From calculations of energies of falling objects on different points, it is clear that potential energy can be transformed into kinetic energy and vice versa, but the total energy of the system remains the same.

Q # 11. State the law of conservation of energy.

Statement

$$Energy\ can\ neither\ be\ created\ nor\ destroyed\ but\ it\ can\ be\ transformed\ from$$
$$one\ kind\ into\ another. The\ total\ amount\ of\ energy\ remains\ constant.$$

EXERCISE SHORT QUESTIONS

Q # 1. A person holds a bag of groceries while standing still, talking to a friend. A car is stationary with the engine running. From the stand point of work, how are these two situations similar?

Ans. In both the above two cases, since there is no displacement, therefore the work done will be zero. Hence in this respect, the two situations are similar.

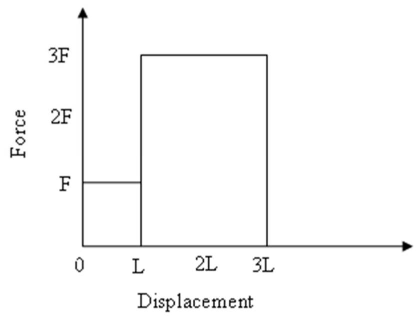
Q # 2. Calculate the work done in kilo joules in lifting a mass of 10 kg (at steady velocity) through a vertical height of 10 m.

Ans. The work done W on the object will be stored in the form of P.E. Therefore:

$$
\begin{aligned}
W &= mgh \\
&= (10)(9.8)(10) \\
&= 980\,J \\
&= 0.98\,kJ
\end{aligned}
$$

Q # 3. A force F acts through a distance L. The force is them increased to 3F, and then acts through a further distance of 2L. Draw the work diagram to scale.

Ans. The force-displacement graph is shown in the figure.



As the work done is equal to the area under the force-displacement curve. Hence

$$
\begin{aligned}
\textit{Work Done} \quad W &= (F \times L) + (3F \times 2L) \\
&= FL + 6FL \\
&= 7FL
\end{aligned}
$$

Q # 4. In which case is more work done? When a 50 kg bag of books is lifted through 50 cm, or when a 50 kg crate is pushed through 2m across the floor with a force of 50 N.

Case 1.

$$
\begin{aligned}
&\textit{Mass} && m = 50\,kg \\
&\textit{Height} && h = 50\,cm = 0.5\,m \\
&\textit{Work} && W = ? \\
&\textit{Work} && W = mgh \\
& && \;\;= (50)(9.8)(0.5) \\
& && \;\;= 245\,J
\end{aligned}
$$

Case 2.

$$
\begin{aligned}
&\textit{Mass} && m = 50\,kg \\
&\text{Distance} && S = 50\,cm = 0.5\,m \\
&\textit{Force} && F = 50\,N \\
&\textit{Work} && W = ? \\
&\textit{Work} && W = FS \\
& && \;\;= (50)(2) \\
& && \;\;= 100\,J
\end{aligned}
$$

Hence in 1st case, more work is done.

Q # 5. An object has 1 J of potential energy. Explain what does it mean?

Ans. It means that work has been dome on the body by the force of 1 N which has lifted the body through a distance of 1 m. This work has been stored in the body in the form of P.E. which is 1J.

Q # 6. A ball of mass m is held at a height h_1 above a table. The table top is at a height h_2 above the floor. One student says that the ball has potential energy mgh_1 but another says that it is $mg(h_1 + h_2)$. Who is correct?

Ans. The 1st student has taken the table as the point of reference for calculating the P.E. While the 2nd student has taken the floor as the point of reference. So both are correct according to their own points of references.

Q # 7. When a rocker re-enters the atmosphere, its nose cone become very hot. Where does this heat energy come from?

Ans. The atmosphere of earth contains a large number of dust particles and water vapors. So when a rocket enters into the atmosphere and passes through these particles, due to the force of friction, the kinetic energy of the rocket is lost in the form of heat. That's why its nose cone becomes very hot.

Q # 8. What sort of energy is in the following:

a) **Compressed spring**
b) **Water in a high dam**
c) **A moving car**

Ans.

a) Elastic Potential Energy
b) Gravitational Potential Energy
c) Kinetic Energy

Q # 9. A girl drops a cup from a certain height, which breaks into pieces. What energy changes are involved?

Ans. When the cup was in the hands of girl, it had gravitational P.E. When the cup is dropped, its P.E. is converted into the K.E. On striking the ground, this energy is converted into sound energy, heat energy and work done in breaking the cup into pieces.

Q # 10. A boy uses a catapult to throw a stone which accidentally smashes a green house window. List the possible energy changes.

Ans. Initially, the catapult had elastic P.E. when the stone is thrown, its P.E. is converted into K.E. On striking the window, this energy is converted into sound energy, heat energy and work dome in breaking the window into pieces.