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CH-4

CONSERVATION OF ENERGY

(i) Conservative & non-conservative forces:

If work done on a system by some forces is depends only on initial and final states of the system and not at all on the paths b/w the two states, they are called the conservative forces. i.e the work done on a system by conservative forces along a closed paths is zero. e.g. Elastic restoring force, Force of gravity.

(ii) The forces whose work along the closed path is not zero are called non conservative forces. e.g. Friction force.

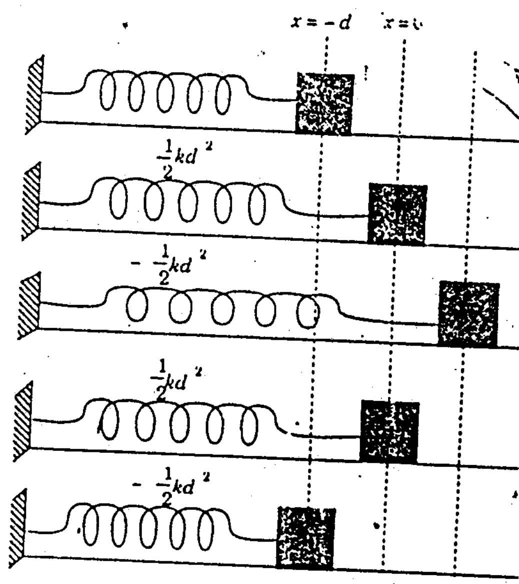
The spring force and gravity force are conservative forces and they give rise to conservative fields. Electrostatic, gravitational and magnetic fields are conservative fields.

To explain the behavior of a conservative system, consider one dimensional motion of a particle acted upon by three separate forces.

I. The spring force $F = -Kx$
II The force of gravity $F = mg$
III The force of friction $F = \mu N$

(i) The Spring Force

Consider a block of mass 'm' attached to a spring having spring constant 'K'. Suppose the body can slide without friction on a horizontal surface. In fig (a) the external agent has compressed the spring from $x=0$ to $x=-d$. When external agent is removed,



The spring begins to do work on the block as it moves from $x=-d$ to $x=0$. The work done by the spring is

$$= \frac{1}{2}k(-d)^2$$

$$= \frac{1}{2}kd^2$$

The work appears as K.E. by work-energy theorem.

In fig (b) the block passes through $x=0$ and the sign of spring force reverses. Now the spring slows down the block, so spring does -ve work.

In fig(c) the block comes to rest for the moment and the work done from $x=0$ to $x=d$ is

$$= -\frac{1}{2}kd^2$$

In fig (d) the spring brings the block back to its mean position, the work done by the spring from $x=d$ to $x=0$ is $= \frac{1}{2}kd^2$

In fig. (e) the spring does -ve work on the block

i.e $= -\frac{1}{2}kd^2$

Total work done on the block by the spring during one complete trip is $= \frac{1}{2}kd^2 - \frac{1}{2}kd^2 + \frac{1}{2}kd^2 - \frac{1}{2}kd^2 = 0$

Such a force under which total work done on a body in a round trip is zero is called conservative force. Hence we conclude that spring force is a conservative force.

It should be noted that the work done by the spring is +ve when it brings the body back to its mean position and work done on the spring (during mean position to extreme position) is -ve

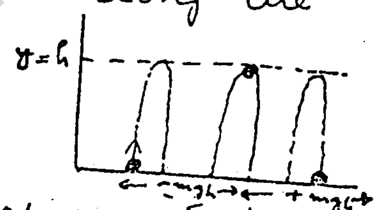
(ii) The Force of Gravity:

Suppose a ball of mass 'm' is projected upward by an external agent with initial velocity v_0. So initial $K.E = \frac{1}{2} m v_0^2$. As the ball rises, the force of gravity does work on it and bring it to rest for a moment at a height $y = h$. The work done by the earth (gravity) on the ball is $= -mgh$ (since the force and displacement are in opposite directions as the ball moves up againt the gravity hence work done is $-ve$).

As the ball falls from $y=h$ to $y=0$ the force of gravity does work $= +mgh$.

Total work done by the gravity-force along the round trip is $= -mgh + mgh$

$= 0$



Thus we conclude that force of gravity is a conservative force.

Alternate Method:

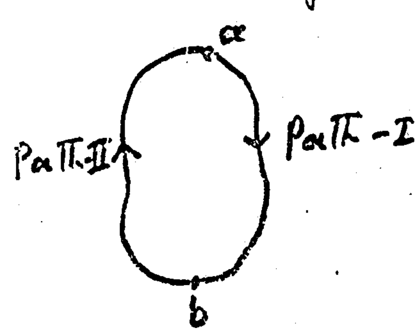
Consider a closed path aba. A praticle moves from 'a' to 'b' and then 'b' to 'a' along two paths ab and ba. If the force acting on the particle is conservative, then work done by the force on the particle along closed path must be zero.

$$W_{Total} = W_{ab} + W_{ba}$$

$$\oint \vec{F}\cdot\vec{ds} = \int_a^b \vec{F}\cdot\vec{ds} + \int_b^a \vec{F}\cdot\vec{ds}$$

$$= \int_a^b \vec{F}\cdot\vec{ds} - \int_a^b \vec{F}\cdot\vec{ds}$$

$$\oint \vec{F}\cdot\vec{ds} = 0$$



Hence the work done by a force in moving a body from initial position to final position is independent of the path taken b/w two points. So the force is conservative.

(iii) **The Frictional Force**

Consider a disc of mass 'm' on the end of a string of length 'R'. The disc is given an initial speed of 'V_0' and it moves in a horizontal circle of radius 'R'. There is a force of friction b/w horizontal surface and the bottom of the disc. The frictional force does −ve work on disk & its direction is opposite to the velocity of disc (at any point).

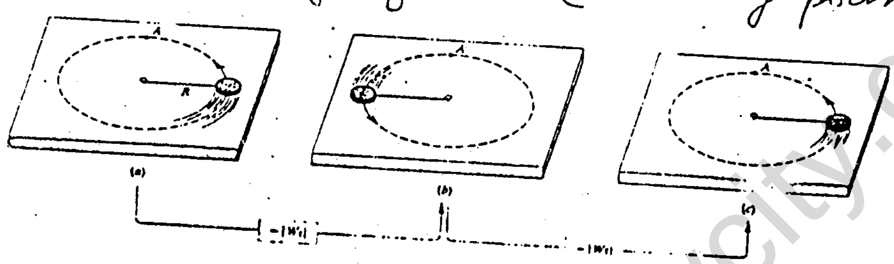


fig.

After the disc has turned to it starting point, the work done by the functional force for the round trip is not zero.

The total work done by friction-force for the round trip $= -W_f - W_f$

$= -2W_f$

The disc returns to its initial position with the smaller K.E after a round trip. So from the above discussion we arrive at the results that

If a body moves under the action of force that does no total work during any round trip, then the force is conservative, otherwise it is non conservative.

The spring force and gravity are the examples of conservative forces while friction-force is an example of non-conservative force.

2. One Dimensional Conservative System

The change in P.E of a particle acted upon by a single conservative force $F(x)$ in one dimension motion in a system is given by;

$$\Delta U = -W \quad \text{———} (1)$$

$$\Delta U = -\int_{x_0}^{x} F(x)\,dx$$

When the particle moves from initial position x_0 to final position x.

Since P.E is a function of position only. So ΔU between x_0 and x is given by;

$$\Delta U = U_{(x)} - U_{(x_0)}$$

$$\Delta U = \int_{x_0}^{x} F(x)\,dx$$

If 'V_0' and 'V' are the velocities of particle at point x_0 and x then according to the work-energy theorem work done by the force is equal to the change in K.E of the particle.

$$W = \Delta K = \int_{x_0}^{x} F(x)\,dx$$

$$W = \int_{x_0}^{x} m\frac{dv}{dt}\,dx$$

$$= \int_{x_0}^{x} m\frac{dv}{dx}\cdot\frac{dx}{dt}\cdot dx$$

$$= \int_{x_0}^{x} mv\,dv$$

$$= \int_{V_0}^{V} m\,V\,dv$$

$$W = \left| \frac{mv^2}{2} \right|_{v_0}^{v}$$

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

$$\therefore \quad -W = \frac{1}{2}mv_0^2 - \frac{1}{2}mv^2 \qquad (2)$$

Now by comparing equ. (1) and equ (2) we get,

$$\Delta U = \frac{1}{2}mv_0^2 - \frac{1}{2}mv^2$$

$$U(x) - U(x_0) = \frac{1}{2}mv_0^2 - \frac{1}{2}mv^2$$

$$U(x) + \frac{1}{2}mv^2 = U(x_0) + \frac{1}{2}mv_0^2 = E \qquad (3)$$

Where 'E' is constant mechanical energy. Equ. (3) is another form of law of conservation of energy for conservative forces.

x———x

Force as a gradiant of P.E.

We can derive the relation b/w force and P.E by the equ.

$$\Delta U = U(x) - U(x_0) = -W$$

$$\Delta U = -\int_{x_0}^{x} F(x)\,dx$$

$$U(x) - U(x_0) = -\int_{x_0}^{x} F(x)\,dx$$

Let $x_0 = \infty$ then $U(x_0) = 0$ so above can be written as

$$U(x) = -\int_{x_0}^{x} F(x)\,dx$$

$$\frac{d\,U(x)}{dx} = -\frac{d}{dx}\int_{0}^{x} F x\,dx$$

$$\frac{d}{dx}U(x) = -F(x)$$

$$\boxed{F(x) = -\frac{d}{dx}U(x)}$$

It means that P.E is a function of position whose -ve derivative gives the force. So the force may be defined as -ve of gradient of P.E.

Examples:

The spring Force:

Suppose the initial position of block is at $x_0 = 0$ i.e the spring is at its mean position so $U(x_0) = 0$ at the mean position.

When we displace the block through a distance x then change in P.E is given by;

$$U(x) - U(x_0) = -W$$

$$= -\int_{x_0}^{x} F(x)\,dx$$

$\because F(x) = -kx$

$$U(x) - 0 = -\int_{x_0}^{x} -kx\,dx$$

$$U(x) = \int_0^x kx\,dx$$

$$U(x) = K\left|\frac{x^2}{2}\right|_0^x$$

$$U(x) = \frac{1}{2}Kx^2 \quad \text{——— (1)}$$

The same result will be obtained when the spring is extended or compressed.

Differentiating equ. (1) w.r.t. x, we have

$$\frac{d}{dx}U(x) = \frac{d}{dx}\frac{1}{2}kx^2$$

$$\frac{d}{dx}U(x) = \frac{1}{2}k(2x)$$

$$\frac{d}{dx}U(x) = kx$$

$$-kx = -\frac{d}{dx}U(x)$$

$\because F = -kx$

$$F = -\frac{d}{dx}U(x)$$

i.e the force is equal to -ve of gradient of P.E.

Suppose the block is moved through a maximum displacement x_m from its mean position, $x_0 = 0$. The P.E stored in the block is $= \frac{1}{2}kx_m^2$.

So when the block is momentarily at rest at extreme position, the entire energy is stored in the form of P.E.

When we release the block from rest, the entire P.E is converted into $K.E_{(max)} = \frac{1}{2}mv_{max}^2$ (At the mean position).

Hence $(P.E)_{max} = (K.E)_{max}$ = Total mechanical Energy

i.e $(P.E)_{max} = (K.E)_{max} = E$

$$\frac{1}{2}kx_{max}^2 = \frac{1}{2}mv_{max}^2 = E$$

or $\frac{1}{2}mv^2 + U(x) = E$

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_{max}^2 \qquad \because U(x) = \frac{1}{2}kx^2$$

$$\frac{1}{2}mv^2 = \frac{1}{2}kx_{max}^2 - \frac{1}{2}kx^2 \qquad \because E = \frac{1}{2}kx_{max}^2.$$

$$mv^2 = kx_{max}^2 - kx^2$$

$$mv^2 = k(x_{max}^2 - x^2)$$

$$v^2 = \frac{k}{m}(x_m^2 \quad x^2)$$

$$\boxed{v = \sqrt{k/m(x_{max}^2 - x^2)}}$$

This is the expression for instantaneous velocity of block.

We find that 'v' depends on 'x'. At $x=0$ i.e at the mean position, the velocity is maximum. Denoting max. velocity with 'v_0' we have

$$v_0 = \sqrt{\frac{K}{m} x_m^2}$$

$$\boxed{v_0 = x_m \sqrt{K/m}}$$

We also find that at $x = x_m$, the velocity is zero.

The Force of Gravity:

Consider the block-earth system. Suppose the refrence point is taken at the surface of the earth i.e $y_0 = 0$ an $U(y_0) = 0$

We now calculate P.E $U(y)$ of the ball-earth system from the relation.

$$U(y) - U(y_0) = -\int_{y_0}^{y} F(y)\,dy$$

Putting $U(y_0) = 0$ and $F(y) = -mg$.

$$U(y) - 0 = -\int_0^y [illegible]$$

$$U(y) = mgy$$

$$\frac{d}{dy} U(y) = -mg \qquad \because F(y) = mg$$

$$F(y) = -\frac{d}{dy} U(y)$$

By the law of conservation of energy along y-axis

$$\frac{1}{2} mv^2 + U(y) = E$$

Putting $U(y) = mgy$

and $E = \frac{1}{2} m v_0^2$ (at max. v)

$$\frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_0^2$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 - mgy$$

$$mv^2 = mv_0^2 - 2mgy$$

$$v^2 = v_0^2 - 2gy$$

From this equation we can find the 'v' at a height 'y'.

x————x

Sample Problem: 1

Sample Problem 1 An elevator cab of mass m (= 920 kg) moves from street level to the top of the World Trade Center in New York, a height h = 412 m above the ground. What is the change in the gravitational potential energy of the cab?

Sol.

$m = 920$ kg

$h = 412$ m

$\Delta U = ?$

As

$$\Delta U = mgh$$

$$= 920 \times 9.8 \times 412$$

$$= 3714592$$

$$= 3.7 \times 10^6 \text{ J}$$

$$\boxed{\Delta U = 3.7 \text{ MJ}}$$ Ans.

Sample Problem: 2

Sample Problem 2 The spring of a spring gun is compressed a distance d of 3.2 cm from its relaxed state, and a ball of mass m (= 12 g) is put in the barrel. With what speed will the ball leave the barrel once the gun is fired? The force constant k of the spring is 7.5 N/cm. Assume no friction and a horizontal gun barrel.

Sol.

$x_0 = d = 3.2 \text{ cm} = 3.2 \times 10^{-2} \text{ m}$

$m = 12 \text{ g} = 12 \times 10^{-3} \text{ kg}$

$K = 7.5 \text{ N/cm} = \dfrac{7.5 \text{ N}}{10^{-2} \text{ m}} = 7.5 \times 10^2 \text{ N/m}$

$V_0 = 0$

As

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = U(x_0) - U(x)$$

$$\frac{1}{2}mv^2 + U(x) = U(x)_0 + \frac{1}{2}mv_0^2$$

$$\frac{1}{2}mv^2 + 0 = \cancel{0+} \frac{1}{2}kd^2 + 0$$

$$\frac{1}{2}mv^2 = \frac{1}{2}Kd^2$$

$$mv^2 = Kd^2$$

$$v^2 = \frac{Kd^2}{m}$$

$$v = \sqrt{\frac{Kd^2}{m}}$$

$$v = d\sqrt{K/m}$$

$$= 3.2\times10^{-2}\sqrt{\frac{7.5\times10^2}{12\times10^{-3}}}$$

$$= 3.2\sqrt{\frac{7.5}{12}\times10^5}\times10^{-2}$$

$$= 3.2\sqrt{\frac{0.75}{12}\times10^6}\times10^{-2}$$

$$= 3.2\times0.25\times10^3\times10^{-2}$$

$$\boxed{v = 8\ m/s}\quad \text{Ans.}$$

Sample Problem: 3

Sample Problem 3 A roller coaster (Fig. 8) slowly lifts a car filled with passengers to a height of $y = 25$ m, from which it accelerates downhill. Neglecting friction in the system, with what speed will the car reach the bottom?

Sol.

$y = 25$ m

There is only one force acting i.e gravity; when the car is at the top, then

$$E_t = U_t + K_t$$

$$= mgy + 0$$

$$= mgy \quad \text{———— (i)}$$

When the car is at the bottom,

$$E_b = U_b + K_b$$

$$= 0 + \frac{1}{2}mv^2$$

$$E_b = \frac{1}{2}mv^2 \quad \text{———— (ii)}$$

By l. law of conservation of energy,

$$E_t = E_b$$

$$\cancel{m}gh = \frac{1}{2}\cancel{m}v^2$$

$$v^2 = 2gh$$

$$v = \sqrt{2gh}$$

$$= \sqrt{2 \times 9.8 \times 25}$$

$$\boxed{v = 22 m/s} \text{ Ans.}$$

x———————x

3. Velocity in terms of 'U' and E.

Stable, Unstable and Neutral Equilibrium

In study of mechanical system, we describe the motion of particle as a function of time. The equ.

$$\frac{1}{2}mv^2 + U(x) = E$$

gives the relation b/w 'v', U, & E for one dimensional motion when force is position dependent.

From the above equ.

$$\frac{1}{2}mv^2 = (E - U)$$

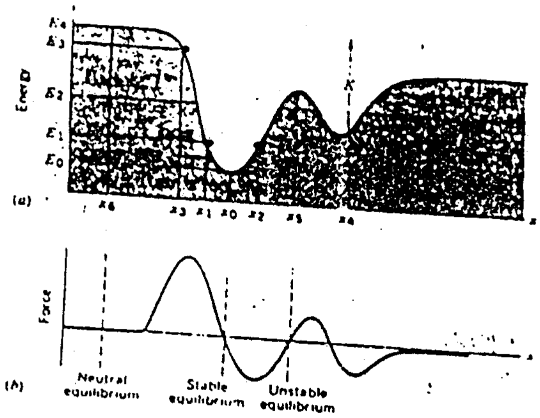
$$v^2 = \frac{2}{m}(E - U)$$

$$v = \sqrt{\frac{2}{m}(E - U)}$$

where 'U' is the P.E and 'E' is the total mechanical energy. From this equ. we find that $v \propto \sqrt{(E - U)}$

Fig (a) shows the variation of energy as a function of distance x.

In fig (a) the lowest mechanical energy is E_0. At this point $E = E_0 = U$ and K.E are zero



So the particle is at rest at x_0. If the system is given slightly higher energy E_1, the particle will move b/w x_1 and x_2. When the particle moves from x_0, its speed decreases at x_1 and x_2. At x_1 and x_2 particle stops and reverses its direction. The points x_1 and x_2 are therefore called the turning points of motion.

At energy E_2 there are four turning points. At energy E_3 there is only one turning point of motion at x_3. At energy E_4, there is no turning point and the particle will not reverse direction.

At a point where $U(x)$ is minimum e.g at $x = x_0$, the slope of curve is zero. So

$$F = \frac{-du}{dx} = 0$$

The particle at rest at that point will remain at rest, the point is called point of stable equilibrium.

Fig (b) $F(x)$ is corresponding to P.E. If particle moves to the left of x_0 then force is +ve and particle is pushed back to x_0.

If the particle move to right of the x_0, the force is −ve and the particle is pushed back to x_0.

At a point where $U(x)$ is max. e.g $x = x_5$, e slope of curve is zero and so

$$F = \frac{-du}{dx} = 0$$

So the particle at rest will remain at rest. However if the particle is displaced from this point the force $F(x)$ will push it away from the equilibrium position. this point is called point of unstable equilibrium. When particle move right to x_5, force is +ve and takes it to larger x.

At the point where $U(x)$ is constant e.g $x = x_6$, the slope curve is zero and

$$F = \frac{-du}{dx} = 0$$

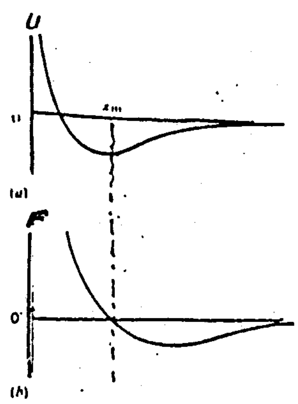
Hence this is called the point of equilibrium.

x ———————————— x

Sample Problem: 4

Sample Problem 4 The potential energy function for the force between two atoms in a diatomic molecule can be expressed approximately as follows:

$$U(x) = \frac{a}{x^{12}} - \frac{b}{x^6},$$



Sol.

$$U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$$

(a) Equilibrium separation b/w the atoms = ?

Equilibrium occurs at $x = x_m$ where $U(x)$ is minimum.

$$\therefore \quad \left(\frac{du}{dx}\right)_{x=x_m} = 0$$

i.e $$\frac{d}{dx}\left(\frac{a}{x^{12}} - \frac{b}{x^6}\right) = 0$$

$$\frac{-12a}{x_m^{13}} + \frac{6b}{x_m^7} = 0$$

$$\frac{6b}{x_m^7} = \frac{12a}{x_m^{13}}$$

$$\frac{x_m^{13}}{x_m^7} = \frac{12a}{6b}$$

$$x_m^6 = 2\frac{a}{b}$$

$$x_m = \left(2\frac{a}{b}\right)^{1/6}$$

(b) The force b/w the atoms = ?

$$F = -\frac{du}{dx}$$

$$= -\frac{d}{dx}\left(\frac{a}{x^{12}} - \frac{b}{x^6}\right)$$

$$F = \frac{12a}{x^{13}} - \frac{6b}{x^7}$$

(c) The minimum energy required to break the molecule apart ie to separate the atoms from equilibrium position to $x = \infty$) = ?

The minimum energy needed to break up the molecule is called dissociation energy E_d.

$$E_d = -U(x_m)$$

$$= -\frac{a}{x_m^{12}} + \frac{b}{x_m^6}$$

$$= \frac{-a}{\left(\left(\frac{2a}{b}\right)^{1/6}\right)^{12}} + \frac{b}{\left(\left(\frac{2a}{b}\right)^{1/6}\right)^{6}} \quad \because x_m = \left(\frac{2a}{b}\right.$$

$$= -a/\left(\frac{2a}{b}\right)^2 + b/(2a/b)$$

$$= \frac{-ab^2}{4a^2} + b^2/2a$$

$$= -b^2/4a + b^2/2a$$

$$E_d = \frac{-b^2 + 2b^2}{4a}$$

$$\boxed{E_d = b^2/4a} \quad \text{Ans.}$$

4. Analytical Solution for $x(t)$.

The function $x(t)$ gives the position 'x' of particle at any time 't'. We readily know that

$$\frac{dx}{dt} = V$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}(E - U(x))} \qquad \because V = \pm\sqrt{\frac{2}{m}(E - U(x))}$$

$$\frac{dx}{\pm\sqrt{\frac{2}{m}(E - U(x))}} = dt$$

Integrating both sides b/w the limits $x_0 \to x$ & $t_0 \to t$. We get

$$\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2}{m}(E - U(x))}} = \int_{t_0}^{t} dt \quad \text{———— (1)}$$

Consider a particle of mass 'm' moving in one dimension and acted upon by a spring of spring constant K. We suppose that at $t=0$, the particle is at extreme position $x = x_0$ and $V = 0$ then

$$U(x) = \frac{1}{2} Kx^2$$

$$\& \quad E = \frac{1}{2} K x_0^2 \quad \text{———— (2)}$$

By putting equ. (2) in expression (1), we get

$$\int_{x_0}^{x} \frac{dx}{\pm\sqrt{\frac{1}{2}Kx_0^2 - \frac{1}{2}Kx^2}} = \int_0^t dt$$

$$\int_{x_0}^{x} \frac{dx}{\pm\sqrt{\frac{K}{m}}\sqrt{x_0^2 - x^2}} = t$$

$$\frac{1}{\pm\sqrt{\frac{K}{m}}} \int_{x_0}^{x} \frac{dx}{\sqrt{x_0^2 - x^2}} = t$$

$$\int_{x_0}^{x} \frac{dx}{\sqrt{x_0^2 - x^2}} = \pm\sqrt{\frac{K}{m}}\, t$$

$$\left| -\cos^{-1}\left(\frac{x}{x_0}\right)\right|_{x_0}^{x} = \pm\sqrt{\frac{K}{m}}\, t \qquad \because \int \frac{dx}{\sqrt{a^2 - x^2}} = -\cos^{-1}\frac{x}{a}$$

$$-\left(\cos^{-1}\frac{x}{x_0} - \cos^{-1}\frac{x_0}{x_0}\right) = \pm\sqrt{\frac{K}{m}}\, t$$

$$\cancel{-}\left(\cos^{-1}\frac{x}{x_0} - 0\right) = \cancel{\pm}\sqrt{\frac{K}{m}}\, t$$

$$\cos^{-1}\frac{x}{x_0} = \sqrt{\frac{K}{m}}\, t$$

$$\frac{x}{x_0} = \cos\left(\sqrt{\frac{K}{m}}\, t\right)$$

$$x = x_0 \cos\left(\sqrt{\frac{K}{m}}\, t\right)$$

$$x(t) = x_0 \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

So one dimensional motion of a particle acted upon by a spring force is oscillating.

Two and Three Dimensional Conservative Systems

The potential energy $U(x)$ for one dimensio-nal system can be modified to three dimensional system as;

$$U(x) = U(x, y, z)$$

then

$$\Delta U = -\int_{x_0}^{x} F(x)\,dx - \int_{y_0}^{y} F(y)\,dy - \int_{z_0}^{z} F(z)\,dz$$

or

$$\Delta U = -\int_{r_0}^{r} \vec{F}(r)\,\vec{dr}$$

where ΔU gives change in potential energy for the system as the particle moves from the point (x_0, y_0, z_0) to with position vector $\vec{r_0}$ to the point (x, y, z) with position vector $\vec{r}$. $F(x)$, $F(y)$, $F(z)$ are the components of conservative force

$$\vec{F(x)} = \vec{F}(x, y, z)$$

For one dimensional motion of particle it is written as

$$\frac{1}{2} mv^2 + U(x) = \frac{1}{2} mv_0^2 + U(x_0)$$

For the three dimensional motion, this expression becomes,

$$\frac{1}{2} mv^2 + U(x, y, z) = \frac{1}{2} mv_0^2 + U(x_0, y_0, z_0).$$

& in vector notation;

$$\frac{1}{2} m\vec{v}.\vec{v} + U(\vec{r}) = \frac{1}{2} m.v_0 \cdot v_0 + U(\vec{r_0})$$

In terms of mechanical energy E, it can be written as;

$$\frac{1}{2} m v^2 + U(x, y, z) = E$$

As we know that

$$F_x = \frac{-du}{dx} = -\frac{\partial U}{\partial x}$$

$$F_y = \frac{-du}{dy} = -\frac{\partial U}{\partial y}$$

$$F_z = \frac{-du}{dz} = -\frac{\partial U}{\partial z}$$

$\therefore$

$$\vec{F}(r) = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

$$= \frac{-\partial U}{\partial x} i - \frac{\partial U}{\partial y} \hat{j} - \frac{\partial U}{\partial z} \hat{k}$$

$$= -\left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} - \frac{\partial}{\partial z} \hat{k} \right) U$$

$$\vec{F}(r) = -\Delta U$$

Thus in the vector language, the conservative force is equal to the -ve of gradient of potential energy.

x———————x

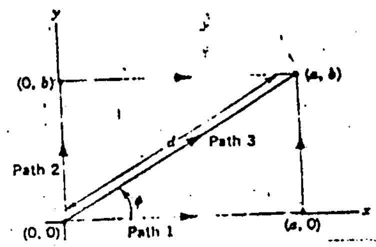
Sample Problem: 5

Sample Problem 5 In a certain system of particles confined to the xy plane, the force has the form $\mathbf{F}(x, y) = F_x\mathbf{i} + F_y\mathbf{j} = -ky\mathbf{i} - kx\mathbf{j}$, where k is a positive constant. (A particle located at an arbitrary point (x, y) is pushed toward the diagonal line $y = -x$ by this force. You can verify this by drawing the line $y = -x$

<u>Solution</u>:

(a) $(\text{Work})_{\text{Path-I}}$ = Work from (0,0) to (a,0) + Work from (a,0) to (a,b)

$$\vec{F} = F_x \hat{i} + F_y \hat{j} = -ky\hat{i} - kx\hat{j} \text{ (given)}$$



$$(\text{Work})_{\text{Path-I}} = \int \vec{F} \cdot \vec{ds} + \int_{y=0}^{y=b} \vec{F} \cdot \vec{ds}$$

$$= \int F_x\, dx + F_y\, dy + \int_{y=0}^{y=b} -Kx\, dy$$

$$= \int F_x dx + \int F_y dy + \int_{y=0}^{y=b} -Ka\, dy$$

$$(\text{Work})_{\text{Path-I}} = \int -Ky\,dx + \int -Kx\,dy - Ka\int_{y=0}^{y=b} dy$$

$$= -K\int y\,dx - K\int x\,dy - Kab$$

$$= -0 - 0 - Kab \qquad \because \text{along } x\text{-axis } y=0 \quad \therefore dy = 0$$

$$\boxed{(\text{Work})_{\text{Path-I}} = -Kab} \quad \text{——— (1)}$$

Now

$$(\text{Work})_{\text{Path-II}} = \text{Work from } (0,0) \text{ to } (0,b) + \text{Work from } (0,b) \text{ to } (a,b)$$

$$(\text{Work})_{\text{Path-II}} = \int \vec{F}\cdot\vec{ds}$$

$$\Rightarrow \int F_x\,dx + F_y\,dy$$

$$= \int F_x\,dx + \int F_y\,dy$$

$$= +\int -Ky\,dx + \int -Kx\,dy$$

$$= \int_{x=0}^{x=a} -Ky\,dx + \int_{y=0}^{y=b} -Kx\,dy$$

Here $y = b$ and $x = 0$ then above expression becomes

$$= \int_{x=0}^{x=a} -Kb\,dx + 0$$

$$= -Kb\,\Big|x\Big|_0^a$$

$$(\text{Work})_{\text{Path-II}} = -Kba = -Kab$$

$$\boxed{(\text{Work})_{\text{Path-II}} = -Kab} \quad \text{——— (2)}$$

Now $$(\text{Work})_{\text{Path-III}} = \int \vec{F}\cdot\vec{ds}$$

$$= \int (F_x\,dx + F_y\,dy)$$

$$= \int (-Ky\,dx - Kx\,dy)$$

$$(\text{Work})_{\text{path III}} = -K \int (y\,dx + x\,dy) \quad \text{——— (3)}$$

Now for from the fig

$$x = r\cos\varphi$$
$$dx = dr\cos\varphi \qquad (\because \cos\varphi \text{ is constant})$$
$$y = r\sin\varphi$$
$$dy = dr\sin\varphi$$

Putting the above values in equ. (3), we get

$$(\text{Work})_{\text{path-III}} = -K \int (y\,dx + \ldots) \quad \text{——— (3)}$$

$$= -K \int r\sin\varphi\, dr\cos\varphi + r\cos\varphi\, dr\sin\varphi$$

$$= -K \int r\,dr \sin\varphi\cos\varphi + r\,dr\sin\varphi\cos\varphi$$

$$= -K \int 2r\,dr\sin\varphi\cos\varphi$$

$$= -2K\sin\varphi\cos\varphi \int_0^d r\,dr \qquad (\text{As } r \text{ is from } 0 \text{ to } d)$$

$$= -2K\sin\varphi\cos\varphi \left| \frac{r^2}{2} \right|_0^d$$

$$= -\cancel{2}K\sin\varphi\cos\varphi \frac{d^2}{\cancel{2}}$$

$$(\text{Work})_{\text{path-III}} = -Kd^2\sin\varphi\cos\varphi$$

Fig. shows that $\cos\varphi = \frac{a}{d}$ & $\sin\varphi = \frac{b}{d}$

hence
$$(\text{Work})_{\text{Path-III}} = -Kd^2\left(\frac{b}{d}\right)\left(\frac{a}{d}\right)$$
$$= -Kd^2\left(\frac{ab}{d^2}\right)$$

$$\boxed{(\text{Work})_{\text{Path-III}} = -Kab} \quad \text{——— (4)}$$

From equ. (1), (2) and (4) we conclude that $W_1 = W_2 = W_3$
Hence work done is independent of the path followed by the body.