ROTATIONAL DYNAMICS

An overview of Rotational Dynamics

For linear motion, dealing with problems of dynamics, we have

Force = Mass * Acceleration

$F = ma$

Now we analyze the rotational motion of rigid bodies about a certain axis of rotation.

Consider a certain force is applied at a certain location to a rigid body free to rotate about a particular axis, the resulting motion depends upon the location of the application of force. A given force applied a one location may produce difference rotation at some other location. This quantity which takes into account both the magnitude of the force and the location and direction in which it is applied is called torque.

We regard force as push or a pull, similarly we can regard torque as twist.

The effort required to put a body into rotation depends on the distribution of mass of the body about the axis of rotation. If the mass is closer to the azis of rotation, it is easier to rotate a body and vice versa. The inertial quantity that takes into account the distribution of body's mass is called rotational inertia or moment of inertia. Unlike mass, rotational inertia is not an intrinsic property of the body.

The equation, identical with equation (1), for rotational dynamics is:

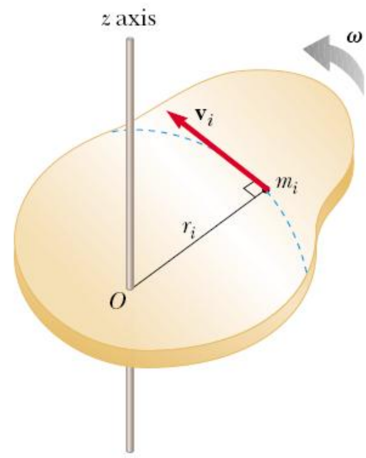
Torque = Moment of Inertia * Angular Acceleration

$\tau = I\alpha$

Relationship between Linear and Angular Variables:

Linear Motion	**Angular Motion**
Linear Displacement x	Angular Displacement ϕ
Linear Velocity $v = \frac{dx}{dt}$	Angular Velocity $\omega = \frac{d\phi}{dt}$
Linear Acceleration $a = \frac{dv}{dt}$	Angular Acceleration $\alpha = \frac{d\omega}{dt}$
Force $F = ma$ or $F = \frac{dp}{dt}$	Torque $\tau = I\alpha$ or $\tau = \frac{dL}{dt}$
Work $W = \int F dx$	Work $W = \int \tau d\phi$
Kinetic Energy $K = \frac{1}{2}mv^2$	Kinetic Energy $K = \frac{1}{2}I\omega^2$
Power $P = F\,v$	Power $P = \tau\,\omega$
Linear Momentum $p = mv$	Angular Momentum $L = I\omega$
Linear Impulse = Ft	Angular Impulse = τt
Mass (Translational Inertial) m	Rotational Inertia I

Kinetic Energy of a Rigid Body



Consider a rigid body is spinning along the axis of rotation with uniform angular velocity ω. Let the object consist of n particles having masses $m_1, m_2, \dots\dots\dots, m_n$, which are at distances $r_1, r_2, \dots\dots\dots, r_n$ from axis of rotation.

The rotational kinetic energy for particle of mass $m_1 = K.E_1 = \frac{1}{2}m_1 r_1^2 \omega_1^2$

The rotational kinetic energy for particle of mass $m_2 = K.E_2 = \frac{1}{2}m_2 r_2^2 \omega_2^2$

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⋮ ⋮ ⋮ ⋮ ⋮

The rotational kinetic energy for particle of mass $m_n = K.E_n = \frac{1}{2}m_n r_n^2 \omega_n^2$

Now the total rotational kinetic energy acting on the rigid body is described as:

$$K.E_{rot} = K.E_1 + K.E_2 + \dots\dots\dots + K.E_n$$

$$\Rightarrow K.E_{rot} = \frac{1}{2}m_1 r_1^2 \omega_1^2 + \frac{1}{2}m_2 r_2^2 \omega_2^2 + \dots\dots\dots + \frac{1}{2}m_n r_n^2 \omega_n^2$$

Since the body is rigid, so all the masses will rotate with same angular velocity ω,

$$\Rightarrow K.E_{rot} = \frac{1}{2}(m_1 r_1^2 + m_2 r_2^2 + \dots\dots\dots + m_n r_n^2)\omega^2$$

$$\Rightarrow K.E_{rot} = \frac{1}{2}\left(\sum_{i=1}^{n} m_i r_i^2\right)\omega^2 = \frac{1}{2}I\omega^2$$

Where $\sum_{i=1}^{n} m_i r_i^2 = I$ is the moment of inertia of the rigid body.

Parallel Axis Theorem

It states that

The rotational inertial of any body about any axis is equal to the sum of rotational inertia about a parallel axis through the centre of mass and the product of mass of body and square of distance between two axis"

Mathematically, it is described as:

$$I = I_{CM} + Md^2$$

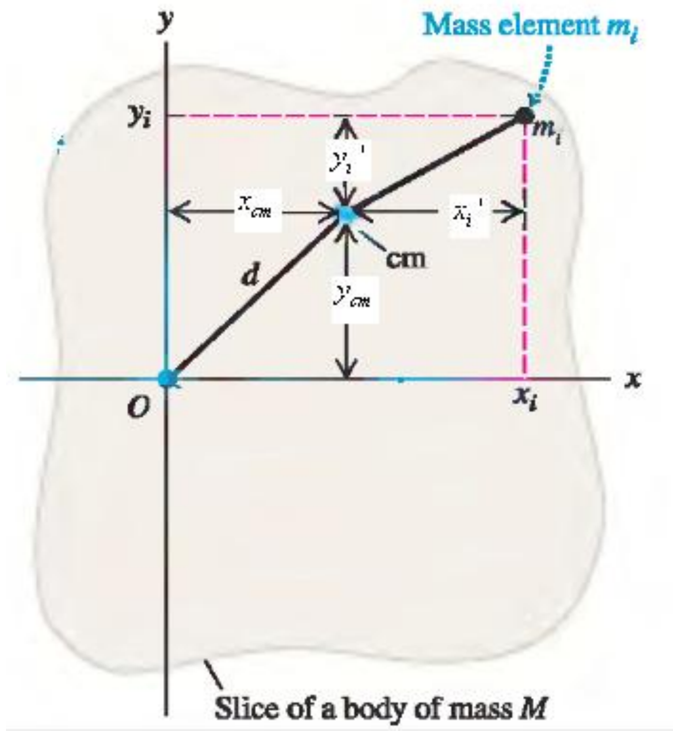
Where

$I =$ Rotational inertia about an arbitrary axis

I_{CM} = Rotational inertia about the parallel axis through the center of mass

M = Mass of the body

d = Perpendicular distance between the axes



Proof

Consider a thin plane slab in xy-plane. The plane can be regarded as the collection of particles each having mass m_i. Now we calculate the rotational inertia of the slab about z-axis passing through origin O and perpendicular to the slab.

(x_i, y_i) = Co-ordinates of m_i w.r.t origin

(x_i', y_i') = Co-ordinates of m_i w.r.t center of mass

(x_{cm}, y_{cm}) = Co-ordinates of center of mass w r t origin

From figure,

$$x_i = x_i' + x_{cm}$$

$$y_i = y_i' + y_{cm}$$

The rotational inertia about the axis through O is

$$I = \sum m_i \left(x_i^2, y_i^2\right)$$

$$I = \sum m_i \left[\left(x_i' + x_{cm}\right)^2 + \left(y_i' + y_{cm}\right)^2\right]$$

$$I = \sum m_i \left[x_i'^2 + x_{cm}^2 + 2x_i' x_{cm} + y_i'^2 + y_{cm}^2 + 2y_i' y_{cm}\right]$$

$$I = \sum m_i \left(x_i'^2 + y_i'^2\right) + \sum m_i \left(x_{cm}^2 + y_{cm}^2\right) + 2x_{cm} \sum m_i x_i' + 2y_{cm} \sum m_i y_i'$$

But $\sum m_i x_i' = 0 = \sum m_i y_i'$

$$x_{cm}^2 + y_{cm}^2 = d^2$$

$$\sum m_i \left(x_i'^2 + y_i'^2\right) = I_{CM}$$

$$\Rightarrow I = I_{CM} + Md^2$$

Which is the mathematical form of parallel axis theorem.

Rotational Inertia for Solid Bodies

If we take very small mass δm tending to zero, then,

$$I = \lim_{\delta m \to 0} \sum r^2\, \delta m$$

$$I = \int r^2\, dm$$

Rotational Inertial of Thin Uniform Rod

Figure shows a thin uniform rod of mass M, length L and area of cross-section A. we want to find out the moment of inertia I about an axis passing through its center O.

$$I = \int r^2\, dm \quad \text{--------------} \quad (1)$$

If ρ is volume mass density, then

$$\rho = \frac{dm}{dv}$$

$$dm = \rho dv$$

$$\Rightarrow dm = \rho A dx \qquad \because dv = Adx$$

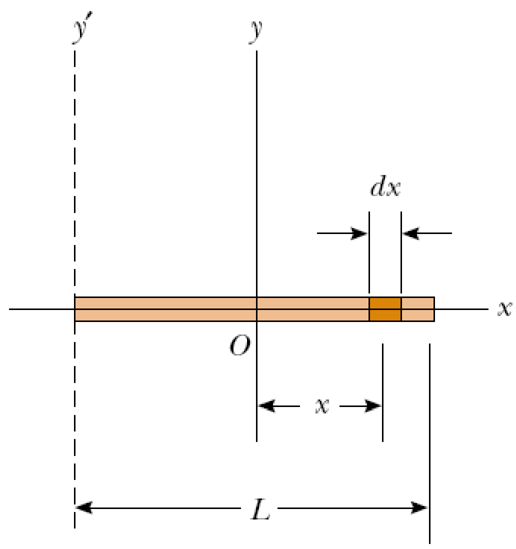
$$\Rightarrow dm = \frac{\rho A L}{L} dx \qquad \because \rho AL = M\ Total\ mass\ of\ the\ rod$$

$$\Rightarrow dm = \frac{M}{L} dx$$

Consider the rod is placed along x-axis, therefore:

$$r = x$$

The equation (1), will become:



$$I = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx$$

$$I = \frac{M}{L} \int_{-L/2}^{L/2} x^2\, dx$$

$$I = \frac{M}{L} \left| \frac{x^3}{3} \right|_{-L/2}^{L/2}$$

$$I = \frac{M}{3L} [(L/2)^3 - (-L/2)^3]$$

$$I = \frac{M}{3L} \left[\frac{L^3}{8} + \frac{L^3}{8} \right]$$

$$I = \frac{M}{3L} \left[\frac{2L^3}{8} \right]$$

$$I = \frac{1}{12} ML^2$$

Rotational Inertia of Thin Uniform Rod about a Perpendicular Axis Passing Through Its Edge

Figure shows a thin uniform rod of mass M, length L and area of cross-section A.
We want to find out the moment of inertia I about a perpendicular axis passing through its edge. As

$$I = \int r^2\, dm \quad \text{--------------} \quad (1)$$

If ρ is volume mass density, then

$$\rho = \frac{dm}{dv}$$

$$dm = \rho dv$$

$$\Rightarrow dm = \rho A dx \qquad \because dv = Adx$$

$$\Rightarrow dm = \frac{\rho AL}{L} dx \qquad \because \rho AL = M \text{ Total mass of the rod}$$

$$\Rightarrow dm = \frac{M}{L} dx$$

Consider the rod is placed along x-axis, therefore:

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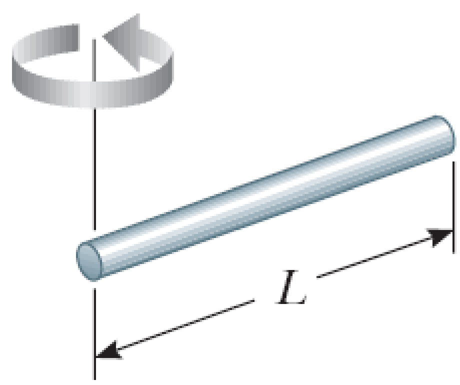


$$I = \int_0^L x^2 . \frac{M}{L} dx$$

$$I = \frac{M}{L} \int_0^L x^2 dx$$

$$I = \frac{M}{L} \left| \frac{x^3}{3} \right|_0^L$$

$$I = \frac{M}{L} \left(\frac{L^3}{3} - 0 \right)$$

$$I = \frac{M}{L} \left(\frac{L^3}{3} \right)$$

$$I = \frac{1}{3} ML^2$$

Rotational Inertia of (i) A Hollow Cylinder (ii) A Solid Cylinder about Axis of Symmetry (Cylindrical Axis)

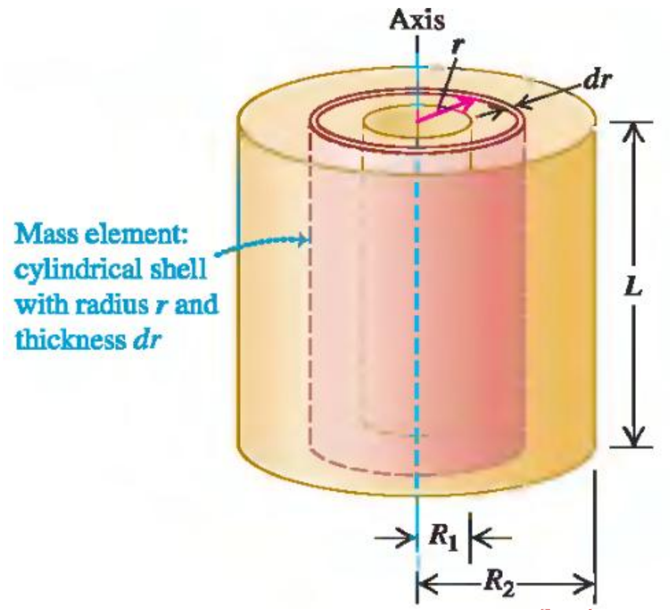
Figure shows annular cylinder of mass M, length L with inner and outer radii R_1 and R_2, respectively. Take a cylindrical shell of radius r and thickness dr.

Surface area of cylindrical shell $ds = 2\pi rL$

Volume of cylindrical shell $dv = 2\pi rLdr$

Mass of cylindrical shell $dm = (2\pi rLdr)\rho$

Rotational inertia of the annular cylinder is



$$I = \int r^2 dm$$

$$I = \int_{R_1}^{R_2} r^2.(2\pi r L dr)\rho$$

$$I = \int_{R_1}^{R_2} 2\pi\rho L\, r^3 dr$$

$$I = 2\pi\rho L \int_{R_1}^{R_2} r^3 dr$$

$$I = 2\pi\rho L \left|\frac{r^4}{4}\right|_{R_1}^{R_2}$$

$$I = \frac{1}{2}\pi\rho L\left(R_2^{\,4} - R_1^{\,4}\right)$$

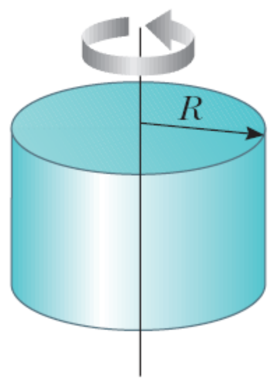
$$I = \frac{1}{2}\pi\rho L\left(R_2^{\,2} - R_1^{\,2}\right)\left(R_2^{\,2} + R_1^{\,2}\right)$$

$$I = \frac{1}{2}\left[\pi\left(R_2^{\,2} - R_1^{\,2}\right)\rho L\right]\left(R_2^{\,2} + R_1^{\,2}\right)$$

Here $M = \pi\left(R_2^{\,2} - R_1^{\,2}\right)\rho L$ = Mass of cylindrical shell

$$I = \frac{1}{2}M\left(R_2^{\,2} + R_1^{\,2}\right) \quad \text{------------} \quad (1)$$

Rotational Inertia of A Solid Cylinder



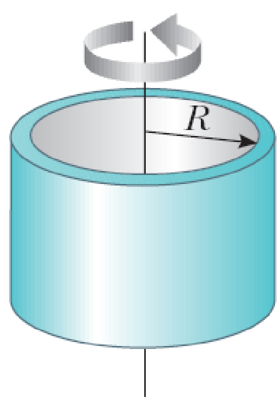
For a solid cylinder, $R_1 = 0$ *and* $R_2 = R$. By putting the values in equation (1), we have:

$$I = \frac{1}{2}M\left(R^2 + 0^2\right)$$

$$I = \frac{1}{2}MR^2$$

Rotational Inertia of A Hollow Cylinder

For a hollow cylinder, $R_1 = R_2 = R$. By putting the values in equation (1), we have:



$$I = \frac{1}{2}M\left(R^2 + R^2\right)$$

$$I = \frac{1}{2}M\left(2R^2\right)$$

$$I = MR^2$$

Rotational Inertia of (i) A Disk (ii) A Hoop (Ring) about Cylindrical Axis

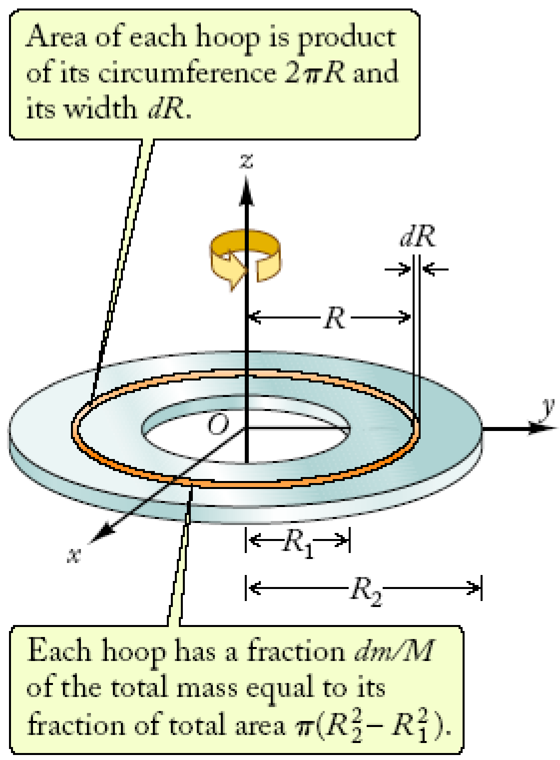
Consider a disk of inner and outer radii R_1 and R_2, respectively. Let σ is surface mass density.

Consider a circular strip of radius r and breadth dr with in the material.

Surface area of circular strip $ds = 2\pi r dr$

Mass of circular strip $ds = \sigma ds = 2\pi r dr \sigma$

Rotational inertia of strip is:



$$I = \int r^2 dm$$

$$I = \int_{R_1}^{R_2} 2\pi\sigma r^3 dr$$

$$I = 2\pi\sigma \int_{R_1}^{R_2} r^3 dr$$

$$I = 2\pi\sigma \int_{R_1}^{R_2} r^3 dr$$

$$I = 2\pi\sigma \left| \frac{r^4}{4} \right|_{R_1}^{R_2}$$

$$I = \frac{1}{2}\pi\sigma\left(R_2^{\ 4} - R_1^{\ 4}\right)$$

$$I = \frac{1}{2}\pi\sigma\left(R_2^{\ 2} - R_1^{\ 2}\right)\left(R_2^{\ 2} + R_1^{\ 2}\right)$$

$$I = \frac{1}{2}\left[\pi\left(R_2^{\ 2} - R_1^{\ 2}\right)\sigma\right]\left(R_2^{\ 2} + R_1^{\ 2}\right)$$

Here $M = \pi\left(R_2^{\ 2} - R_1^{\ 2}\right)\sigma$ = Mass of cylindrical shell

$$I = \frac{1}{2}M\left(R_2^{\ 2} + R_1^{\ 2}\right) \quad \text{------------} \quad (1)$$

Rotational Inertia of A Solid Disk

For a solid disk, $R_1 = 0 \; and \; R_2 = R$. By putting the values in equation (1), we have:

$$I = \frac{1}{2}M\left(R^2 + 0^2\right)$$

$$I = \frac{1}{2}MR^2$$

Rotational Inertia of A Hollow Cylinder

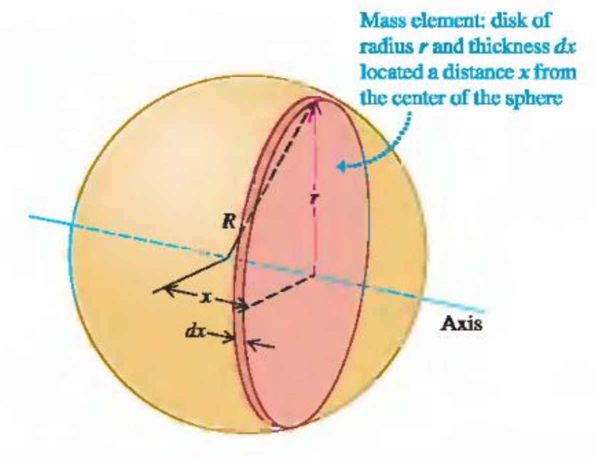
For a hoop or a ring, $R_1 = R_2 = R$. By putting the values in equation (1), we have:

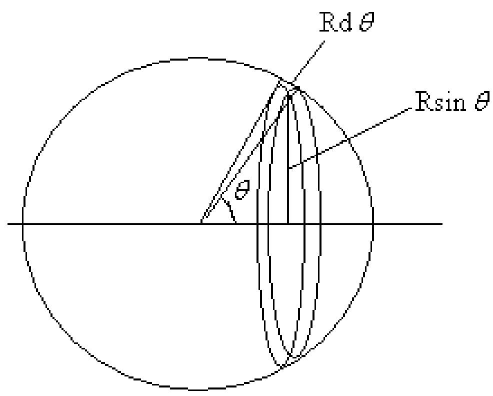
$$I = \frac{1}{2}M\left(R^2 + R^2\right)$$

$$I = MR^2$$

Rotational inertia of a spherical shell

Fig. shows a thin spherical shell of radius R, mass m and surface mass density σ. In fig R$\sin\theta$ is the radius of the circular ring, dθ angular width and Rdθ linear width of the ring.





Circumference of the ring = $2\pi R$

Surface area of the circular strip ds =$(2\pi R\ \sin\theta)(\text{R}d\theta) = 2\pi R^2\ \sin\theta\, d\theta$

Mass of circular strip dm = σ ds= $2\pi R^2\ \sigma\ \sin\theta\, d\theta$

Rotational inertia of the circular strip about the diameter of the shell

$$\text{dI} = r^2 dm = (R\sin\theta)^2\ 2\pi R^2\ \sigma\ \sin\theta\, d\theta$$

$$\text{dI} = 2\pi\ \sigma\ \ R^4 \sin^3\theta d\theta$$

Rotational inertia of the shell is

$$I = \int dI = \int r^2 dm = \int_0^\pi 2\pi\ \sigma\ \ R^4\sin^3\theta d\theta$$

$$= 2\pi\ \sigma\ \ R^4 \int_0^\pi \sin^3\theta\, d\theta \quad \text{----------} \quad (1)$$

Here

$$\int_0^\pi \sin^3\theta d\theta = \int_0^\pi \sin^2\theta \sin\theta = \int_0^\pi (1-\cos^2\theta)\sin\theta d\theta$$

$$= \int_0^\pi (\sin\theta + \cos^2\theta(-\sin\theta))d\theta$$

$$= \int_0^\pi \sin\theta d\theta + \int_0^\pi \cos^2\theta(-\sin\theta)d\theta$$

$$= \ \left|-\cos\theta\right|_0^\pi + \left|\frac{\cos\theta}{3}\right|_0^\pi$$

$$= (-\cos\pi - (-\cos 0)) + \left(-\frac{1}{3} - \frac{1}{3}\right) = 2 - \frac{2}{3} = \frac{4}{3}$$

Hence equation (1) will becomes:

$$I = 2\pi\ \sigma\ R^4 . \frac{4}{3} = \frac{2}{3}(4\pi R^2 \sigma) R^2$$

$$I = \frac{2}{3} MR^2 \qquad \text{where } M = (4\pi R^2 \sigma)$$

Rotational inertia of a solid sphere about its diameter

Fig, shows a solid sphere of radius R, mass M volume mass density ρ.Take a spherical shell of radius r and width dr with in sphere.

$$dI = \frac{2}{3} r^2 * \text{mass} = \frac{2}{3} r^2 * dm$$

$$\frac{mass\ of\ shell}{mass\ of\ sphere} = \frac{volume\ of\ shell}{volume\ of\ sphere}$$

$$\frac{dm}{M} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi R^3}$$

$$\text{dm} = \frac{3r^2}{R^3} Mdr$$

So

$$dI = \frac{2}{3} r^2 * \frac{3r^2}{R^3} Mdr$$

$$dI = \frac{2M}{R^3} r^4 dr$$

The moment of inertia of the sphere is the sum of moment of inertia of the various shells.

$$I = \int dI = \frac{2M}{R^3} \int_0^R r^4 dr$$

$$= \frac{2M}{R^3}\left[\frac{R^5}{5} - 0\right] = \frac{2}{5} MR^2$$

Hence

$$I = \frac{2}{5} MR^2$$