CONSERVATION OF ENERGY

We come across two types of forces in our daily life:

- Conservative forces
- Non-conservative forces

Conservative Forces

If the work done by the force on the body depends upon the initial and final locations and is independent of path taken by the body between the two points, then such a force is a conservative force.

Or

If the net work done by the force on the body along the close path is zero, then such a force is called conservative force.

Conservative forces are also distinguished by the ability to store energy only due configuration of the system. This stored energy is called potential energy.

Examples of Conservative Forces

The Spring Force

Consider a block of mass m attached to a spring of spring constant k. The block is capable of moving on horizontal frictionless table.

Figure 1. The external agent has displaced the object from mean position (x = 0) to extreme position (x = +A).

Figure 2. The external agent is suddenly removed at $t = 0$ and the spring begins to do work on the block.

As the object moves from x = +A to x = 0, the spring does work $W_1 = \frac{1}{2}kA^2$.

Figure 3. When the spring moves from $x = 0$ to $x = -A$, the spring force reverses and spring acts to slow down the block and does negative work. $W_2 = -\frac{1}{2}kA^2$

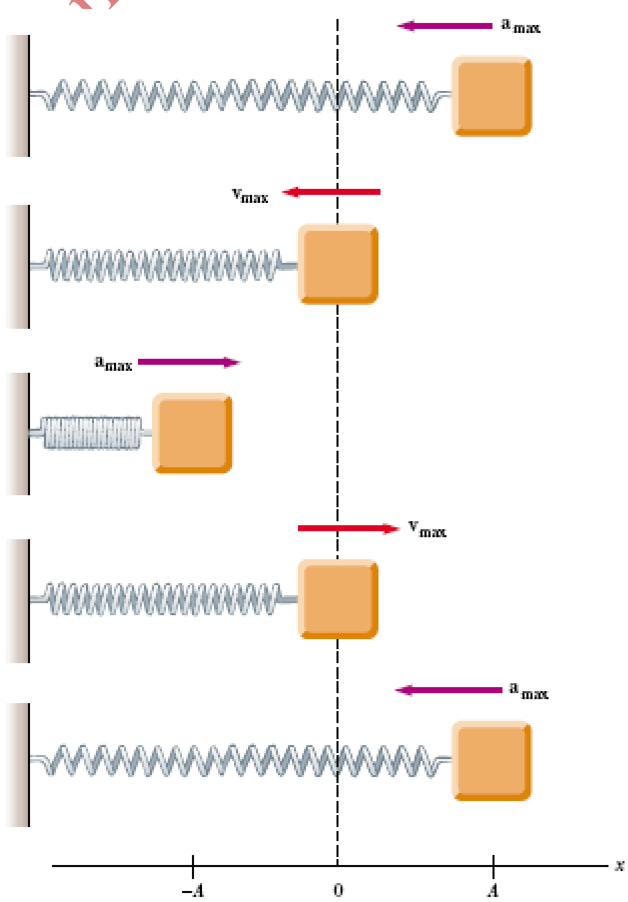


Figure 4. The object moves from spring moves $x = -A$ to $x = 0$, and spring does work $W_3 = \frac{1}{2}kA^2$.

Figure 5. The object moves from $x=0$ to $x=+A$. The spring does negative work $W_2=-\frac{1}{2}kA^2$ (as the block slows down).

The total work done by the spring force during the complete cycle (close path) is zero:

Total work $W_T = W_1 + W_2 + W_3 + W_4$

$$W_T = \frac{1}{2}kA^2 + \left(-\frac{1}{2}kA^2\right) + \frac{1}{2}kA^2 + \left(-\frac{1}{2}kA^2\right) = 0$$

So, the spring force is conservative force.

Spring Force is Independent of Path Followed

Consider of a mass spring system. The block moves from $x=+A$ to $x=-\frac{A}{2}$ along two different paths.

Path 1. The block moves directly from $x=+A$ to $x=-\frac{A}{2}$. The work done by the spring:

$$W_1 = \int_{+A}^{-\frac{A}{2}} (-kx)\,dx = \left|-\frac{1}{2}kx^2\right|_{+A}^{-\frac{A}{2}} = -\frac{1}{2}k\left[\left(-\frac{A}{2}\right)^2 - A^2\right]$$

$$\Rightarrow W_1 = -\frac{1}{2}k\left[\frac{A^2}{4} - A^2\right] = -\frac{1}{2}k\left[-\frac{3A^2}{4}\right]$$

$$\Rightarrow W_1 = -\frac{3kA^2}{8}$$

Path 2. From $x=+A$ to $x=-A$ and from $x=-A$ to $x=-\frac{A}{2}$

$$W_2 = \int_{+A}^{-A} (-kx)\,dx + \int_{-A}^{-\frac{A}{2}} (-kx)\,dx$$

$$W_2 = \left|-\frac{1}{2}kx^2\right|_{+A}^{-A} + \left|-\frac{1}{2}kx^2\right|_{-A}^{-\frac{A}{2}}$$

$$\Rightarrow W_2 = -\frac{1}{2}k\left[(-A)^2 - A^2\right] - \frac{1}{2}k\left[\left(-\frac{A}{2}\right)^2 - (-A)^2\right]$$

$$\Rightarrow W_2 = -\frac{1}{2}k[0] - \frac{1}{2}k\left[\frac{A^2}{4} - A^2\right]$$

$$\Rightarrow W_2 = -\frac{1}{2}k\left[-\frac{3A^2}{4}\right]$$

$$\Rightarrow W_2 = -\frac{3kA^2}{8}$$

Conclusion: As $W_1 = W_2$, so work done is independent of path followed.

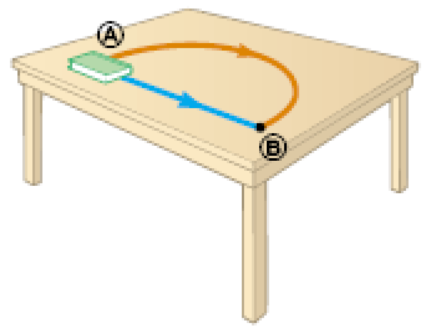
The Force of Gravity

Consider a ball of mass m is thrown upwards by an external agent. As the ball rises to height $y=h$, the force of gravity mg acts downwards while the distance covered is upwards. So the work done by the earth on the body is $-mgh$. The ball momentarily comes to rest and then falls from $y=h$ to $y=0$, the work done by the gravity is $+mgh$.

The total work done along the round trip $=-mgh+mgh=0$. So the force of gravity or gravitational force is a conservative force.

From the criterion that work done on a system by a conservative force along a closed path is zero, we have:

$$\oint \vec{F}.d\vec{s} = \underset{path1}{\int_A^B} \vec{F}.d\vec{s} + \underset{path2}{\int_B^A} \vec{F}.d\vec{s} = 0$$



$$\underset{path1}{\int_A^B} \vec{F}.d\vec{s} = -\underset{path2}{\int_B^A} \vec{F}.d\vec{s}$$

$$\underset{path1}{\int_A^B} \vec{F}.d\vec{s} = \underset{path2}{\int_A^B} \vec{F}.d\vec{s}$$

This shows that the work done on a system by a conservative force between two points is independent of path.

**Electrostatic and Magnetic forces are conservative forces, while the frictional force is an example of a non-conservative force.*

Potential Energy

Potential energy is the energy of the configuration of the system and can only be defined for conservative forces such as spring force or the force of gravity.

When configuration of a system undergoes a change, work is done by the conservative force. Thus

$\Delta U = -W$

The change in potential in a process is equal to the negative of work done by the conservative force.

If E is the mechanical energy of a conservative system, then by law of conservation of energy:

$U + K = E$ (constant)

Or $\Delta(U+K)=0$

Where U and V are the potential energy and kinetic energy of the mechanical system.

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One Dimensional Conservative System

When a one dimensional conservative force F(x) acts upon a body and as the result the object moves from x_0 to x, then the change in P.E. is given by the expression:

$$\Delta U = -W = -\int_{x_0}^{x} F(x)\,dx \qquad \text{----------} \quad (1)$$

Equation (1) can also be described as:

$$U(x) - U(x_0) = -\int_{x_0}^{x} F(x)\,dx \qquad \text{----------} \quad (2)$$

If x_0 is an arbitrary reference point, then potential energy function ΔU can be obtained. If x_0 is at infinity: $U(x_0) = U(\infty) = 0$, then the resulting function $U(x)$ can be used to calculate the P.E. at the particular points in motion, say x_1 and x_2.

In moving from x_0 to x, the velocity of the particle changes from v_0 to v. Hence according to the Work Energy Theorem, the work done by the force is:

$$W = \Delta K = \frac{1}{2}mv^2 - \frac{1}{2}m{v_0}^2$$

From equation (1),

$$\Delta U = -W$$

$$\Rightarrow U(x) - U(x_0) = -\left(\frac{1}{2}mv^2 - \frac{1}{2}m{v_0}^2\right)$$

$$\Rightarrow U(x) - U(x_0) = -\frac{1}{2}mv^2 + \frac{1}{2}m{v_0}^2$$

$$\Rightarrow U(x) + \frac{1}{2}mv^2 = U(x_0) + \frac{1}{2}m{v_0}^2$$

$$\Rightarrow U(x) + \frac{1}{2}mv^2 = E \qquad \text{----------} \quad (3)$$

Where E is the mechanical energy of the system which depends upon depends upon the initial position x_0 and initial velocity v_0; which have the definite values. So it is constant during motion i.e., the mechanical energy E is constant.

Differentiating equation (2), we have:

$$\frac{d}{dx}\left[U(x) - U(x_0)\right] = -\frac{d}{dx}\int_{x_0}^{x} F(x)\,dx = -\int_{x_0}^{x} \frac{d}{dx}F(x)\,dx$$

$$\frac{dU(x)}{dx} = -F(x)$$ $\frac{dU(x_0)}{dx} = 0$ as $U(x_0)$ is a constant.

$$F(x) = \frac{dU(x)}{dx}$$

Thus

"*Potential energy is a function of position whose negative derivative gives force*".

The Spring Force

Consider a mass-spring system. When the block is displaced a distance x from x_0, the potential energy of the system is given by:

$$U(x)-U(x_0)=-\int_{x_0}^{x}F(x)\,dx$$

Initially, the mass is at mean position i.e., $x_0=0$, so $U(x_0)=0$. And $F(x)=-kx$

$$U(x)=-\int_{x_0}^{x}(-kx)\,dx$$

$$U(x)=\frac{1}{2}kx^2 \quad \text{----------} \quad (1)$$

When the spring is compressed or stretched (i.e. either x is negative or positive), we have same results.

Differentiating equation (1), we have:

$$\frac{dU(x)}{dx}=\frac{d}{dx}\left(\frac{1}{2}kx^2\right)=kx$$

$$-\frac{dU(x)}{dx}=-kx$$

$$-\frac{dU(x)}{dx}=F(x)$$

Suppose the block is stretched from $x=0$ to a distance x_m, then the energy stored in the mass spring system is $\frac{1}{2}kx_m{}^2$. As the block is momentarily at rest, so its $\text{K.E}=0$. Hence the entire energy is its P.E.

$$E=\frac{1}{2}kx_m{}^2$$

From this, the instantaneous velocity is:

$$v=\sqrt{\frac{k}{m}(x_m{}^2-x^2)}$$

The speed of the bock is maximum at $x=x_0=0$: $v_{\max}=v_0=\sqrt{\frac{k}{m}}x_m$

The speed of the bock is zero at $x=x_m$: $v=0$

The Gravitational Force

For the ball-earth system, we choose $y_0 = 0$ as a reference point at the surface of the earth, therefore $U(y_0) = 0$

Using relation

$$U(y) - U(y_0) = -\int_{y_0}^{y} F(y)\,dy$$

Putting the values $U(y_0) = 0$ and $F(y) = -mg$

$$U(y) = -\int_{y_0}^{y} (-mg)\,dy$$

$$U(y) = mgy$$

Differentiating, we get:

$$\frac{dU(y)}{dy} = \frac{d(mgy)}{dy} = mg$$

$$-\frac{dU(y)}{dy} = -mg = F(y)$$

$$F(y) = -\frac{dU(y)}{dy}$$

Using the law of conservation of mechanical energy (in y-direction), we have:

$$\frac{1}{2}mv^2 + U(y) = E$$

Putting the values $U(y) = mgy$ and $E = \frac{1}{2}mv_0{}^2$:

$$\frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_0{}^2$$

$$v^2 + 2gy = v_0{}^2$$

$$v^2 = v_0{}^2 - 2gy$$

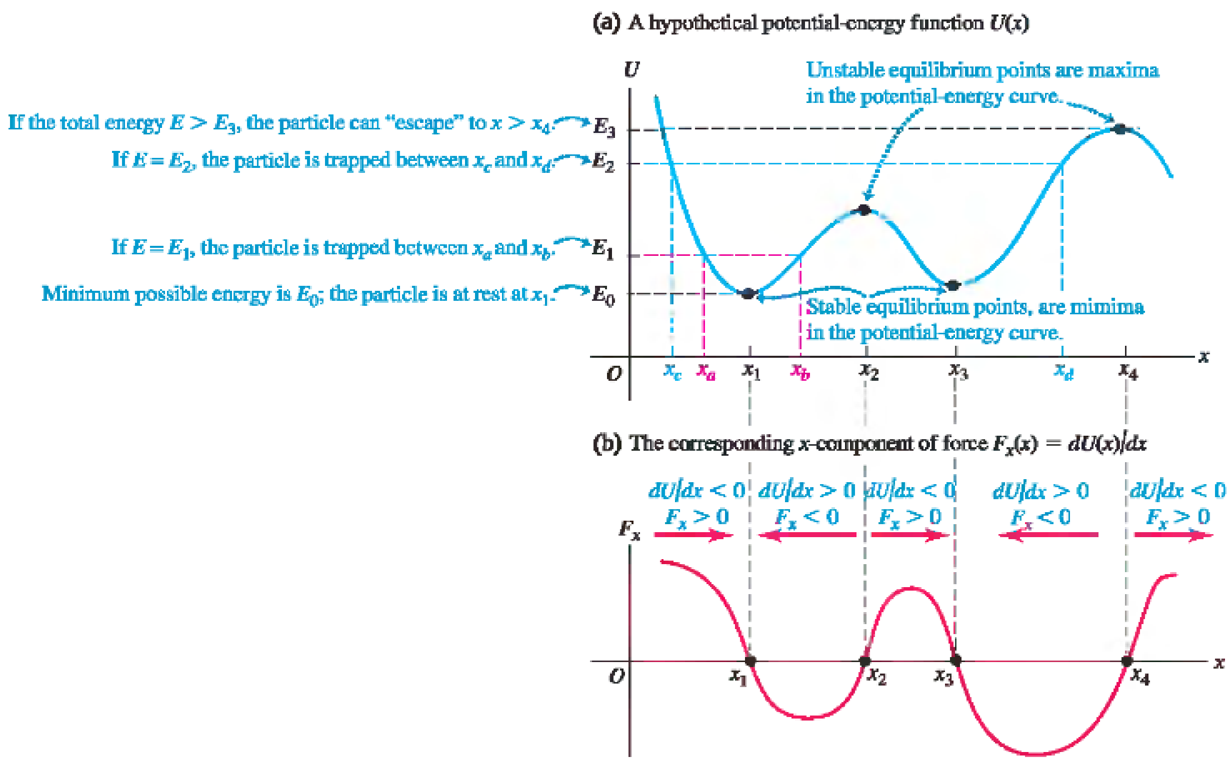
This is the expression of velocity of block at height y in earth's gravitational field.

One Dimensional Conservative System (Energy Method)

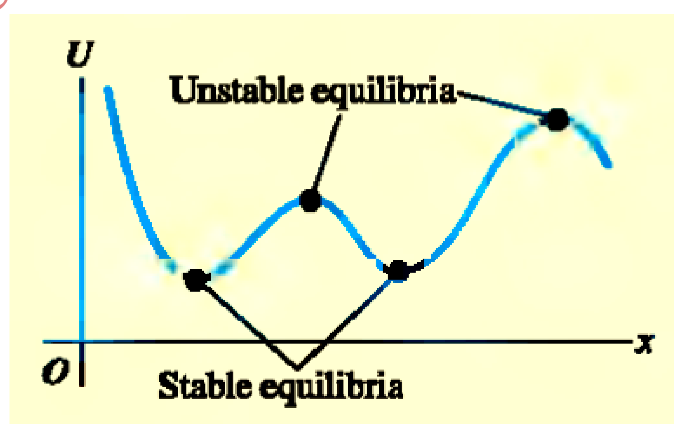
At each point, the force F_x on the any object is equal to the negative of the slope of the U(x) curve:

$$F_x = -\frac{dU}{dx}$$

Points x_1 and x_3 are stable equilibrium points. At each of these points, F_x is zero because the slope of the U(x) curve is zero. When the particle is displaced to either side, the force pushes back toward the equilibrium point. An analogous situation is a marble rolling around in a round-bottomed bowl. We say that at points x_1 and x_3 is a point of stable equilibrium. More generally, any minimum in a potential-energy curve is a stable equilibrium position.



The slope of the U(x) curve is also zero at points x_2 and x_4, and these are also equilibrium points. But when the particle is displaced a little to the right of either point, the slope of the U(x) curve becomes negative, corresponding to a positive F_x that tends to push the particle still farther from the point. When the particle is displaced a little to the left, F_x is negative, again pushing away from equilibrium. This is analogous to a marble rolling on the top of a bowling ball. Points x_2 and x_4 are called unstable equilibrium points; any maximum in a potential-energy curve is an unstable.



the direction of a conservative force The direction of the force on a body is not determined by the sign of the potential energy U. Rather, it's the sign of $F_x = -\frac{dU}{dx}$ that matters.

If the total energy is E_1 and the particle is initially near x_1, it can move only in the region between x_a and x_b determined by the intersection of the E_1 and U graphs (Fig. a). Again, U cannot be greater than E) because K can't be negative. We speak of the particle as moving in a potential well, and x_a and x_b are the turning points of the particle's motion (since at these points, the particle stops and reverses direction). If we increase the total energy to the level E_2, the particle can move over a wider range, from x_c and x_d. If the total energy is greater than E_3, the particle can "escape" and move to indefinitely large values of x. At the other extreme, E_o represents the least possible total energy the system can have.

Sample problem 4.

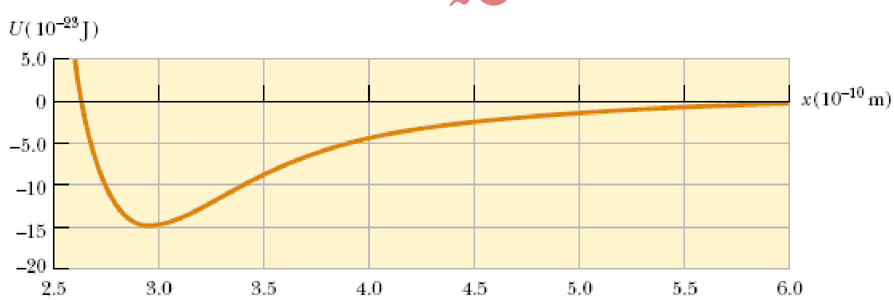
The potential energy function for the force between two atoms in a diatomic molecule can be expressed approximately as:

$$U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$$

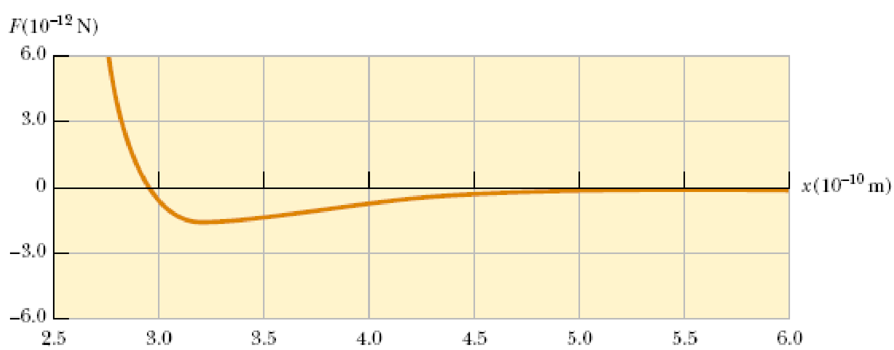
Where a and b are the positive constants and x is the distance between the atoms. Find (a) the equilibrium separation between the atoms. (b) The force between atoms and (c) minimum energy necessary to break the molecule apart (i.e., to separate the atoms fron equilibrium position to $x = \infty$).

Solution

The plot of potential energy $U(x)$ verses separation x is shown in the figure a and the plot of force F verses x as shown in the figure b.



(a)



(b)

Let the equilibrium occurs between the atoms at $x = x_m$ at which $U(x)$ is minimum and is found from:

$$\left(\frac{dU}{dx}\right)_{x=x_m} = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{a}{x^{12}}-\frac{b}{x^{6}}\right)=0$$

$$\Rightarrow \left(\frac{-12a}{x^{13}}-\frac{(-6)b}{x^{7}}\right)=\left(\frac{-12a}{x^{13}}+\frac{6b}{x^{7}}\right)=0$$

Multiplying above equation by $\frac{x^7}{6}$:

$$\left(\frac{-2a}{x^{6}}+\frac{b}{1}\right)=0$$

$$\Rightarrow \frac{2a}{x^{6}}=b \Rightarrow x^{6}=\frac{2a}{b}$$

$$x_m=\left(\frac{2a}{b}\right)^{\frac{1}{6}}$$

The is the require expression for equilibrium separation between the atoms of diatomic molecule.

(b). The force corresponding to the minimum potential energy is:

$$F=-\frac{dU}{dx}=-\frac{d}{dx}\left(\frac{a}{x^{12}}-\frac{b}{x^{6}}\right)=\frac{12a}{x^{13}}-\frac{6b}{x^{7}}$$

This force is positive between $x=0$ and $x=x_m$, the atoms are repelled from one another and the force is directed towards increasing x.. When the force is negative from $x=x_m$ and $x=\infty$, the atoms are attractive to one another (this force is directed towards decreasing x).

At $x=x_m$, the force is zero. This is the equilibrium point and is stable equilibrium.

(c). The minimum energy required to break the molecule into separate atoms is called dissociation energy E_d we can separate the atoms to $x=\infty$ when $U=0$, whenever $E\geq 0$. The minimum energy required corresponds to $E=0$ i.e., the atoms are infinitely separated ($U=0$). The energy added to the molecule in its equilibrium state to rise its energy from negative value to zero is called its dissociation energy E_d.

$$U(x_m)+E_d=0$$

$$E_d=-U(x_m)=-\frac{a}{{x_m}^{12}}+\frac{b}{{x_m}^{6}}$$

Substituting $x_m=\left(\frac{2a}{b}\right)^{\frac{1}{6}}$, we get:

$$E_d=\frac{b^2}{4a}$$

Two and Three Dimensional Conservative Systems

In three dimensions, the potential energy can be written as $U(x, y, z)$. So the equation

$\Delta U = U(x) - U(x_0) = -\int_{x_0}^{x} F(x)\, dx$ in three dimensions is

$$\Delta U = -\int_{x_0}^{x} F_x\, dx - \int_{y_0}^{y} F_y\, dy - \int_{z_0}^{z} F_z\, dz$$

Where ΔU is the change in potential energy for the system when the particle moves from point (x_0, y_0, z_0) to (x, y, z). F_x, F_y and F_z are the components of conservative force $F(r) = F(x, y, z)$.

The relation $\frac{1}{2}mv^2 + U(x) = \frac{1}{2}mv_0{}^2 + U(x_0)$ in three dimensional motion is:

$$\frac{1}{2}mv^2 + U(x, y, z) = \frac{1}{2}mv_0{}^2 + U(x_0, y_0, z_0)$$

In vector notation:

$$\frac{1}{2}m\vec{v}.\vec{v} + U(\vec{r}) = \frac{1}{2}m\vec{v}_0.\vec{v}_0 + U(\vec{r}_0)$$

Where $\vec{v}.\vec{v} = v_x{}^2 + v_y{}^2 + v_z{}^2$ and $\vec{v}_0.\vec{v}_0 = v_{0x}{}^2 + v_{0y}{}^2 + v_{0z}{}^2$

In terms of mechanical energy:

$$\frac{1}{2}mv^2 + U(x, y, z) = E$$

Also

$$F_x = -\frac{dU}{dx} = -\frac{\delta U}{\delta x}$$

$$F_y = -\frac{dU}{dy} = -\frac{\delta U}{\delta y}$$

$$F_z = -\frac{dU}{dz} = -\frac{\delta U}{\delta z}$$

Putting values in equation

$$\vec{F}(r) = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$\vec{F}(r) = -\frac{\delta U}{\delta x}\hat{i} - \frac{\delta U}{\delta y}\hat{j} - \frac{\delta U}{\delta z}\hat{k}$$

$$\vec{F}(r) = -\left(\frac{\delta}{\delta x}\hat{i} - \frac{\delta}{\delta y}\hat{j} - \frac{\delta}{\delta z}\hat{k}\right)U$$

$$\vec{F}(r) = -\vec{\nabla}U$$

So, the conservative force is equal to the negative of gradient of potential energy $U(x, y, z)$.

Conservation of Energy in System of Particles

Energy can be transformed from one kind to another in an isolated system, but it cannot be created or destroyed: the total energy of the system remains constant.

Explanation

Consider a block-spring system. The block is placed on a table and a frictional force is present between the block and the table. There are two transfer of energy through the system boundary: the positive conservative work W_s done on the block by the spring and the negative work W_f done on the block by the frictional force exerted by the table. For this system, the conservation of energy is written as