THE ELECTRIC FIELD

28.1 Electric Field

The region or space around a charge in which it can exert the force of attraction or repulsion on other charged bodies is called electric field.

28.2 Electric Field Intensity

The electrostatic force on unit positive charge at a specific field point is called the electric field intensity. In order to find out electric field intensity due a point charge 'q', a test charge 'q_0' is placed in its electric field at a field point. The electric field intensity **E** due to a point charge 'q' is expressed as,

$$\mathbf{E} = \frac{\mathbf{F}}{q_0} \qquad \text{----------------------} \qquad (1)$$

where **F** is the electrostatic force between the source charge 'q' and test charge 'q_0'.

The test charge 'q_0' should be very small, so that it cannot disturb the field produced by source charge 'q'. Therefore the electric field intensity can be written as,

$$\mathbf{E} = \lim_{q_0 \to 0} \frac{\mathbf{F}}{q_0} \qquad \text{----------------------} \qquad (2)$$

28.3 Electric Field Intensity Due To a Point Charge

Consider a test charge 'q_0' placed at point P in the electric field of a point charge 'q' at a distance 'r' apart.

We want to find out electric field intensity at point 'P' due to a point charge 'q'.

The electrostatic force 'F' between 'q' and 'q_0' can be find out by using expression,

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q q_0}{r^2} \qquad \text{----------------------} \qquad (3)$$

The electric field intensity 'E' due to a point charge 'q' can be obtained by putting the value of electrostatic force in equation (1),

$$E = \frac{\left(\frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\right)}{q_0}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \text{----------------------- (4)}$$

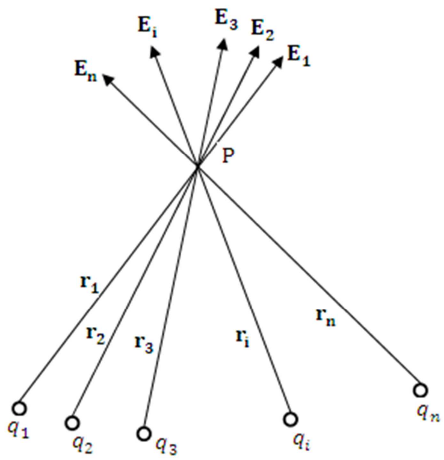
This expression gives the magnitude of electric field intensity due to a point charge 'q'. In vector form, the electric field intensity '**E**' will be:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} \qquad \text{---------------------- (5)}$$

where $\hat{r}$ is the unit vector which gives the direction of electric field intensity.

28.4 Electric Field due to Many (*n*) Point Charge

Let $q_1, q_2, q_3, \ldots\ldots\ldots, q_n$ are the 'n' point charges as shown in the figure.



The electric field intensity due to assembly of 'n' point charges at a specific field point can be determined by following the procedure mentioned below;

- Calculate the electric field intensity at a field point due to each charge separately by assuming that the other charges are absent.
- Calculate the total electric field intensity by taking the vector sum of intensities of individual point charges.

Now if $\mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}, \ldots\ldots\ldots, \mathbf{E_n}$ be the electric field intensities at a field point due to the point charges $q_1, q_2, q_3, \ldots\ldots\ldots, q_n$ respectively. Then, the total electric field intensity due to assembly of 'n' point charges will be;

$$\mathbf{E} = \mathbf{E_1} + \mathbf{E_2} + \mathbf{E_3} + \ldots\ldots + \mathbf{E_n} \qquad \text{---------------------- (6)}$$

where

$\mathbf{E_1}$ = Electric Field Intensity at a Field Point due to Point Charge 'q_1' = $\frac{1}{4\pi\varepsilon_0}\frac{q_1}{{r_1}^2}\widehat{r_1}$

$\mathbf{E_2}$ = Electric Field Intensity at a Field Point due to Point Charge 'q_2' = $\dfrac{1}{4\pi\varepsilon_0}\dfrac{q_2}{{r_2}^2}\widehat{r_2}$

$\mathbf{E_3}$ = Electric Field Intensity at a Field Point due to Point Charge 'q_3' = $\dfrac{1}{4\pi\varepsilon_0}\dfrac{q_3}{{r_3}^2}\widehat{r_3}$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$\mathbf{E_n}$ = Electric Field Intensity at a Field Point due to Point Charge 'q_n' = $\dfrac{1}{4\pi\varepsilon_0}\dfrac{q_n}{{r_n}^2}\widehat{r_n}$

Putting values in equation (6), we get

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{{r_1}^2}\widehat{r_1} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{{r_2}^2}\widehat{r_2} + \frac{1}{4\pi\varepsilon_0}\frac{q_3}{{r_3}^2}\widehat{r_3} + \ldots\ldots\ldots + \frac{1}{4\pi\varepsilon_0}\frac{q_n}{{r_n}^2}\widehat{r_n}$$

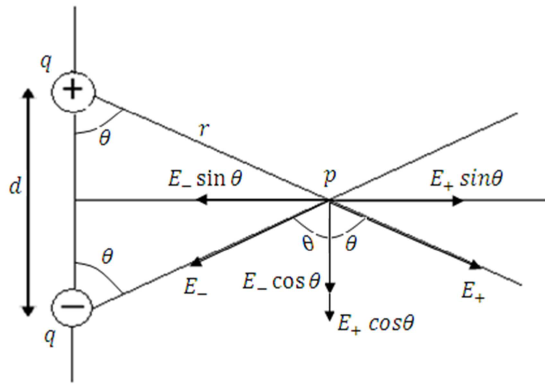
$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{{r_1}^2}\widehat{r_1} + \frac{q_2}{{r_2}^2}\widehat{r_2} + \frac{q_3}{{r_3}^2}\widehat{r_3} + \ldots\ldots\ldots + \frac{q_n}{{r_n}^2}\widehat{r_n}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{{r_i}^2}\widehat{r_i} \qquad \text{----------------------} \ \mathbf{(7)}$$

This equation gives the total electric field intensity due to assembly of 'n' point charges at a specific field point.

28.5 Electric Field due to a Dipole

Two equal and opposite charges separated by a small distance form an electrical dipole. Consider two point charges $+q$ and $-q$ of equal magnitude lying distance 'd' apart as shown in the figure below.



We want to determine the electric field intensity **E** due to a dipole at point 'P'. The point 'P' is at a distance 'x' along the perpendicular bisector of the line joining the charges. Let the electric field intensities at point P due to the charges $+q$ and $-q$ be E_+ and E_- respectively.

The total electric field intensity at point P due to the charges $+q$ and $-q$ is given by the expression.

$$E = E_+ + E_- \qquad \text{----------------------} \qquad \textbf{(8)}$$

For the present case, $E_+ = E_-$ because the point P is equidistant for the charges $+q$ and $-q$. Therefore,

$$E_+ = E_- = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{\left[x^2+\left(\frac{d}{2}\right)^2\right]} \qquad \text{----------------------} \qquad \textbf{(9)}$$

From the figure it is clear that the y −components of E_+ and E_- will cancel the effect of each other while the x −components of E_+ and E_- added up to give of resultant electric field intensity of electric dipole. Therefore,

$$E = E_+ \cos\theta + E_- \cos\theta$$

$$\therefore E_+ = E_-$$

$$E = 2E_+ \cos\theta \qquad \text{----------------------} \qquad \textbf{(10)}$$

From figure,

$$\cos\theta = \frac{d/2}{r}$$

$$\cos\theta = \frac{d/2}{\sqrt{x^2+\left(\frac{d}{2}\right)^2}}$$

By putting the value of E_+ and $\cos\theta$ in equation (10), we get

$$E = 2E_+\left(\frac{d/2}{\sqrt{x^2+\left(\frac{d}{2}\right)^2}}\right) = E_+\left(\frac{d}{\sqrt{x^2+\left(\frac{d}{2}\right)^2}}\right)$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{\left[x^2+\left(\frac{d}{2}\right)^2\right]}\left(\frac{d}{\sqrt{x^2+\left(\frac{d}{2}\right)^2}}\right)$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{qd}{\left[x^2+\left(\frac{d}{2}\right)^2\right]^{\frac{3}{2}}}$$

The quantity $p = qd$ is called dipole moment. Therefore,

$$E = \frac{1}{4\pi\varepsilon_0}\frac{p}{\left[x^2+\left(\frac{d}{2}\right)^2\right]^{\frac{3}{2}}}$$

$$E = \frac{1}{4\pi\varepsilon_0} \frac{p}{x^3\left[1+\frac{1}{x^2}\left(\frac{d}{2}\right)^2\right]^{\frac{3}{2}}}$$

$$E = \frac{1}{4\pi\varepsilon_0} \frac{p}{x^3}\left[1+\left(\frac{d}{2x}\right)^2\right]^{-\frac{3}{2}}$$

By using binomial expansion, we get

$$E = \frac{1}{4\pi\varepsilon_0} \frac{p}{x^3}\left[1+\left(-\frac{3}{2}\right)\left(\frac{d}{2x}\right)^2+\cdots\cdots\cdots\right]$$

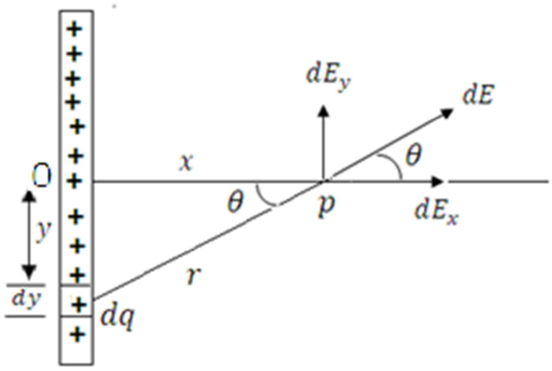
If we neglect the 2nd and higher terms, then

$$E = \frac{1}{4\pi\varepsilon_0} \frac{p}{x^3} \qquad \text{----------------------} \qquad \textbf{(11)}$$

This is the expression of electric field due to a dipole.

28.6 Electric Field Intensity due to an Infinite Line of Charges

Consider an infinite line of positive charge as shown in the figure below.



We want to calculate the electric field intensity at point 'P' at a perpendicular distance 'x' from the line of charge. As the charge is distributed uniformly over it, so it has constant linear charge density μ;

$$\mu = \frac{Charge}{Length} = \frac{q}{L}$$

For an infinitesimal length element 'dy' having charge 'dq'

$$\mu = \frac{dq}{dy}$$

$$dq = \mu\, dy$$

The electric field intensity due to this length element at point 'P' is given by;

$$dE = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0} \frac{\mu\, dy}{r^2}$$

$$dE = \frac{1}{4\pi\varepsilon_0} \frac{\mu\, dy}{x^2+y^2} \qquad \text{----------------------} \qquad \textbf{(12)}$$

The rectangular components of 'dE' are

$dE_x = dE\ cos\theta$ and $dE_y = dE\ sin\theta$

If we consider identical charge elements symmetrically located on both sides of 'O', then the y −components of electrical field intensity will cancel out each other and its x −components are added up to give the total electric field intensity due to this continuous charge distribution. Therefore, the total electric field intensity will be

$$E = E_x = \int_{y=-\infty}^{y=\infty} dE_x$$

$$= \int_{y=-\infty}^{y=\infty} dE\ cos\theta$$

$$= 2\int_{y=0}^{y=\infty} dE\ cos\theta$$

Putting value of 'dE' from equation (12), we get

$$E = 2\int_{y=0}^{y=\infty} \frac{1}{4\pi\varepsilon_0} \frac{\mu\, dy}{x^2 + y^2} cos\theta$$

$$E = \frac{1}{2\pi\varepsilon_0} \int_{y=0}^{y=\infty} \frac{\mu\, dy}{x^2 + y^2} cos\theta \text{ ---------------------- } \textbf{(13)}$$

From figure,

$$\frac{y}{x} = tan\theta$$

$$y = x\ tan\theta$$

$$dy = x\ sec^2\theta d\theta$$

Putting values in equation (13), we get

$$E = \frac{\mu}{2\pi\varepsilon_0} \int_{y=0}^{y=\infty} \frac{x\ sec^2\theta d\theta}{x^2 + y^2} cos\theta$$

$$E = \frac{\mu}{2\pi\varepsilon_0} \int_{y=0}^{y=\infty} \frac{x\ sec\theta d\theta}{x^2\left(1 + \frac{y^2}{x^2}\right)}$$

$$\therefore \frac{y}{x} = tan\theta$$

$$E = \frac{\mu}{2\pi\varepsilon_0} \int_{y=0}^{y=\infty} \frac{sec\theta d\theta}{x(1 + tan^2\theta)}$$

$$E = \frac{\mu}{2\pi\varepsilon_0 x} \int_{y=0}^{y=\infty} \frac{sec\theta d\theta}{sec^2\theta}$$

$$E = \frac{\mu}{2\pi\varepsilon_0 x} \int_{y=0}^{y=\infty} cos\theta d\theta$$

$$\therefore considering\ \frac{y}{x} = tan\theta$$

$$\therefore when\ y = 0,\ \theta = 0$$

$$\therefore when\ y = \infty,\ \theta = \frac{\pi}{2}$$

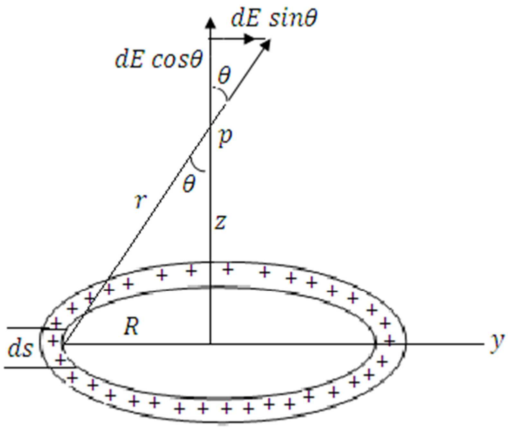
Therefore,

$$E = \frac{\mu}{2\pi\varepsilon_0 x} \int_{\theta=0}^{\theta=\frac{\pi}{2}} cos\theta d\theta$$
$$E = \frac{\mu}{2\pi\varepsilon_0 x} |sin\theta|_0^{\pi/2}$$
$$E = \frac{\mu}{2\pi\varepsilon_0 x} \left(sin\frac{\pi}{2} - \sin 0\right)$$
$$E = \frac{\mu}{2\pi\varepsilon_0 x} (1-0)$$
$$E = \frac{\mu}{2\pi\varepsilon_0 x}$$

This is the expression for electric field intensity at point 'P' due to an infinite line of charge.

28.7 Electric Field Intensity due to a Ring of Charge

Consider a ring of positive charge of radius 'R' as shown in the figure below.



We want to find out the electric field intensity at point 'P' which is at the distance 'z' from the plane of ring.

As the charge is distributed uniformly over it, so it has constant linear charge density μ. For an infinitesimal length element 'ds' of ring,

$$\mu = \frac{dq}{ds}$$
$$dq = \mu\, ds$$

The electric field intensity due to the charge dq at point 'P' is given by;

$$dE = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0} \frac{\mu\, ds}{r^2}$$
$$dE = \frac{1}{4\pi\varepsilon_0} \frac{\mu\, ds}{z^2 + R^2} \qquad \text{----------------------} \qquad \textbf{(14)}$$

The rectangular components of 'dE' are

$dE_z = dE\ cos\theta$ and $dE_y = dE\ sin\theta$

If we consider the identical charge elements located on the opposite end of tne diameter, then dE_y components will cancel out each other and dE_z components are added up to give the final value of electric field intensity at point P. Therefore, the total electric field intensity will be

$$E = E_z = \int dE_z$$

$$= \int dE \cos\theta$$

Putting value of 'dE' from equation (12), we get

$$E = \frac{1}{4\pi\varepsilon_0} \int \frac{\mu\, ds}{z^2 + R^2} \cos\theta$$

From figure, $\cos\theta = \frac{z}{R}$

$$E = \frac{1}{4\pi\varepsilon_0} \int \frac{\mu\, ds}{z^2 + R^2} \cdot \frac{z}{r} \qquad \text{----------------------} \quad \textbf{(15)}$$

From figure, $z^2 + R^2 = r^2$

$$E = \frac{\mu z}{4\pi\varepsilon_0} \int \frac{ds}{r^3}$$

$$E = \frac{\mu z}{4\pi\varepsilon_0 r^3} \int ds$$

$$E = \frac{\mu z}{4\pi\varepsilon_0 r^3} (2\pi r)$$

$$E = \frac{(2\pi r\mu) z}{4\pi\varepsilon_0 (z^2 + R^2)^{3/2}}$$

$$E = \frac{qz}{4\pi\varepsilon_0 (z^2 + R^2)^{3/2}}$$

$$\therefore \mu = \frac{total\ charge}{total\ length} = \frac{q}{2\pi r}$$

$$\therefore 2\pi r\,\mu = q$$

When the point P is far away from ring, i.e., $z \gg R$ so that R^2 can be neglected.

$$E = \frac{qz}{4\pi\varepsilon_0 (z^2)^{3/2}}$$

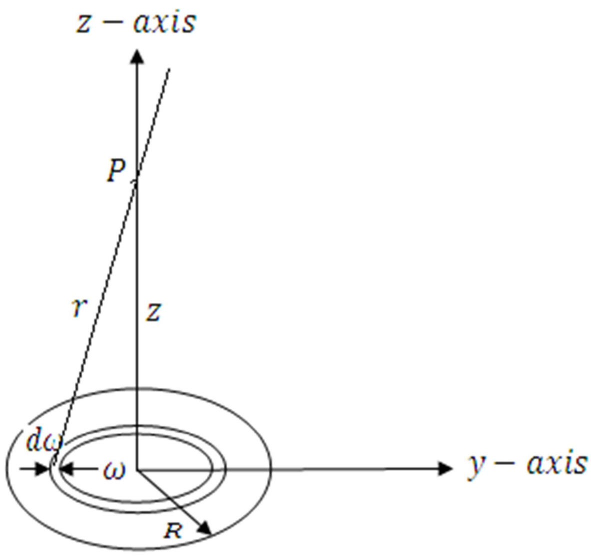
$$E = \frac{qz}{4\pi\varepsilon_0 z^3}$$

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{z^2}$$

Which is the expression for the electric field intensity, when the field point is far away from the ring. Thus, the charged ring acts like a point charge when the field point is at the large distance.

28.8 Electric Field Intensity due to a Disk of Charge

Consider a circular disk of uniform surface charge density as shown in the figure below. We want to find out the electric field intensity at point 'P' which is at the distance 'z' from the plane of disk.



Consider a small element of the disk in the ring shape of radius 'ω' and width '$d\omega$'. If 'dq' is the charge on this element of ring, then

$$\sigma = \frac{dq}{dA}$$

$\therefore$ *where dA is the is the area of length element*

$$dq = \sigma\, dA$$

$$dq = \sigma\,(2\pi\omega\, d\omega)$$

$\therefore dA = 2\pi\omega\, d\omega$

We know the electric field intensity due to the ring of charge is given by;

$$E = \frac{qz}{4\pi\varepsilon_0 (z^2 + \omega^2)^{3/2}} \qquad \text{----------------------} \qquad \textbf{(16)}$$

Consider the ring as the differential part of the disk, the equation (16) becomes

$$dE = \frac{z\, dq}{4\pi\varepsilon_0 (z^2 + \omega^2)^{3/2}}$$

By putting the value of 'dq', we get

$$dE = \frac{z\,\sigma\,(2\pi\omega\, d\omega)}{4\pi\varepsilon_0 (z^2 + \omega^2)^{3/2}}$$

$$= \frac{z\,\sigma}{4\varepsilon_0}\ \frac{(2\omega\, d\omega)}{(z^2 + \omega^2)^{3/2}}$$

$$= \frac{z\,\sigma}{4\varepsilon_0}\ (z^2 + \omega^2)^{-3/2}\,(2\omega\, d\omega)$$

Now total electric field intensity at point P due to whole disk can be calculated integrating the above expression from '$\omega = 0$ to $\omega = R$'

$$E = \int_{\omega=0}^{\omega=R} dE$$

$$E = \frac{z\,\sigma}{4\varepsilon_0} \int_{\omega=0}^{\omega=R} (z^2 + \omega^2)^{-3/2} (2\omega\, d\omega)$$

$$E = \frac{z\,\sigma}{4\varepsilon_0} \left| \frac{(z^2 + \omega^2)^{\frac{-3}{2}+1}}{\frac{-3}{2}+1} \right|_0^R$$

$$E = \frac{z\,\sigma}{4\varepsilon_0} \left| \frac{(z^2 + \omega^2)^{\frac{-1}{2}}}{\frac{-1}{2}} \right|_0^R$$

$$E = -\frac{z\,\sigma}{2\varepsilon_0} \left| \frac{1}{\sqrt{z^2 + \omega^2}} \right|_0^R$$

$$E = -\frac{z\,\sigma}{2\varepsilon_0} \left(\frac{1}{\sqrt{z^2 + R^2}} - \frac{1}{\sqrt{z^2 + 0}} \right)$$

$$E = -\frac{z\,\sigma}{2\varepsilon_0} \left(\frac{1}{\sqrt{z^2 + R^2}} - \frac{1}{z} \right)$$

$$E = \frac{z\,\sigma}{2\varepsilon_0} \left(\frac{1}{z} - \frac{1}{\sqrt{z^2 + R^2}} \right)$$

$$E = \frac{\sigma}{2\varepsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + R^2}} \right)$$

If $R \gg z$; i.e., $z = 0$

Then $\frac{z}{\sqrt{z^2 + R^2}} = \frac{0}{\sqrt{0^2 + R^2}} = 0$

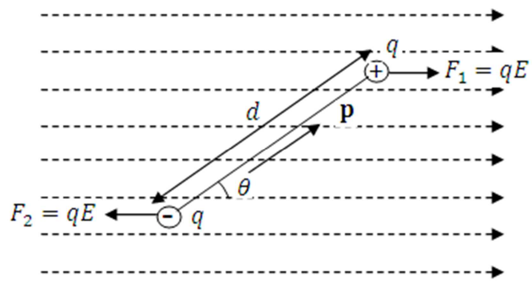
Hence

$$E = \frac{\sigma}{2\varepsilon_0}$$

It means when $z = 0$, the disk of charge behaves like infinite sheet of charge.

28.9 Torque on a Dipole in a Uniform Electric Field

Consider an electric dipole consists of $+q$ and $-q$ separated by a distance d is shown in the figure below.



When an electric dipole is placed in an external electric field, the force on the positive charge will be in one direction and the force on the negative charge in another direction. The force on $+q$ and $-q$ have the equal magnitude but opposite in direction. Therefore the net force on the dipole due to external field is zero. The magnitude of each force is

$$F_1 = F_2 = qE$$

These two forces make a couple and so the torque acts on the dipole. The magnitude of this torque is given as

Torque = (Force) (Moment Arm)

$$\tau = F\,(d\,sin\theta)$$

$$\tau = qEd\,sin\theta$$

$$\tau = pE\,sin\theta$$

$$\therefore Dipole\ Moment\ p = qd$$

In vector form,

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} \qquad \text{----------------------} \qquad \textbf{(17)}$$

The direction of torque of dipole in a uniform electric field is determined by right hand rule.

28.10 Energy of Dipole in a Uniform Electric Field

Consider an electrical dipole is placed in a uniform electric field. We want to calculate the work done by the electric field in turning the dipole through an angle ‘θ’. The work done by the electric field in turning the dipole from an initial angle θ_i to the final angle θ_f is given by;

$$W = \int dw$$

$$W = -\int_{\theta_i}^{\theta_f} \tau\, d\theta$$

Where 'τ' is the torque exerted by the electric field. The negative sign is necessary because the torque tends to decrease θ. Also

$$\tau = pE\, sin\theta$$

Therefore,

$$W = -\int_{\theta_i}^{\theta_f} pE\, sin\theta\, d\theta$$

Assuming that the dipole is revolve from initial angle $\theta_i = \frac{\pi}{2}$ to the final angle $\theta_f = \theta$. Then

$$W = \int_{\frac{\pi}{2}}^{\theta} pE\, sin\theta\, d\theta$$

$$W = pE \int_{\frac{\pi}{2}}^{\theta} sin\theta\, d\theta$$

$$W = pE\, |cos\theta\, |_{\pi/2}^{\theta}$$

$$W = pE\, [\, cos\theta - cos\frac{\pi}{2}\,]$$

$$W = pE\, cos\theta$$

Since the work done by the agent that produces the external field is equal to the negative of the change in potential energy U of the system.

$$U = -W = -pE\, cos\theta$$

Therefore

Potential Energy $U = -\mathbf{p}.\mathbf{E}$ ---------------------- **(18)**

This is the expression of potential energy of the dipole in a uniform electric field.

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