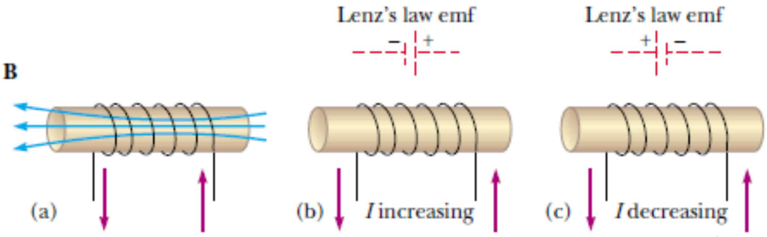
INDUCTANCE

Inductance

Consider a coil wound on a cylindrical core. Assume that the current in the coil either increases or decreases with time. When the current flow through the coil, a magnetic field will be set up inside the coil as shown in figure below.



As the current changes with time, the magnetic flux through the coil also changes and induces an emf in the coil. This induced emf ε_L is directly proportional to the rate of change of current $\frac{dI}{dt}$.

$$\varepsilon_L \propto \frac{dI}{dt}$$

$$\varepsilon_L = -L\frac{dI}{dt}$$

The negative sign is due to the fact that induced emf oppose the cause which produce it. The symbol L is the inductance or self inductance of the coil which is described as:

"The ratio between the induced emf and the rate of change of current"

$$L = -\frac{\varepsilon_L}{\left(\frac{dI}{dt}\right)}$$

The unit of inductance is Henry. The inductance of a coil is one Henry if an emf of one volt is induced in it, when the current changes at the rate of one ampere per second.

$$1\,Henry = \frac{1\,volt}{\left(\frac{1\,ampere}{second}\right)} = 1\,VA^{-1}s$$

Inductance of a Solenoid

Consider a solenoid of length l, having N number of turns. If the current I flows through solenoid, then magnetic field B produced is given by:

$$B = \frac{\mu_0 N\,I}{l} = \mu_0 n\,I$$

Where $n = \frac{N}{l}$ is the number of turns per unit length.

If A is the area of each circular turn of the solenoid, then the flux ϕ_B passing through one turn is:

$$\phi_B = (B)(A)$$

$$\phi_B = \mu_0 n\,IA$$

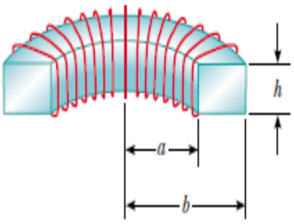
The magnetic flux passing through all the N turns is given by:

$$\phi_B = N\mu_0 n\,IA$$

If the current changes through the solenoid, then the magnetic flux also changes with respect to the time, i.e.,

$$\frac{d\phi_B}{dt} = N\mu_0 n\,A\frac{dI}{dt}$$

According to the Faraday's law of electromagnetism, the induced emf ε is described as:



$$\varepsilon = -\frac{d\phi_B}{dt} = -N\mu_0 n\, A\frac{dI}{dt} \qquad \text{----------------} \quad \textbf{(1)}$$

If L is the self inductance of the solenoid, then

$$\varepsilon \quad = -L\,\frac{dI}{dt} \qquad \text{----------------} \quad \textbf{(2)}$$

Comparing (1) and (2), we get:

$$L = N\mu_0 n\, A$$

$$\because As\; n = \frac{N}{l} \Longrightarrow N = nl$$

$$\Longrightarrow L = (nl)\mu_0 n\, A$$

$$\Longrightarrow L = \mu_0 n^2 l\, A$$

This expression gives the self inductance of a solenoid.

Inductance of a toroid

Let N is the number of turns in a toroid through current I is flowing and r is mean radius of the toroid. Let a and b are the inner and outer radii of toroid, respectively, and h is its height.

The magnetic field produced at any point inside toroid can be find out using expression:

$$B = \frac{\mu_0 NI}{2\pi r}$$

Let $h\; dr = dA$ is the small area element, then the magnetic flux passing through the small area element will be:

$$d\phi_B = B\; dA$$

$$d\phi_B = B\; h\; dr$$

Integrating we get:

$$\phi_B = \int_{r=a}^{r=b} B\; h\; dr$$

The magnetic flux passing through all N turns is:

$$\phi_B = \int_{r=a}^{r=b} N\; B\; h\; dr$$

$$\phi_B = \int_{r=a}^{r=b} N\left(\frac{\mu_0 NI}{2\pi r}\right) h\; dr = \frac{\mu_0 N^2 I\; h}{2\pi}\int_{r=a}^{r=b}\frac{dr}{r}$$

$$\phi_B = \frac{\mu_0 N^2 I\; h}{2\pi}\ln|r|_a^b = \frac{\mu_0 N^2 I\; h}{2\pi}(\ln b - \ln a)$$

$$\phi_B = \frac{\mu_0 N^2 I\; h}{2\pi}\;\ln\frac{b}{a}$$

$$\frac{\phi_B}{I} = \frac{\mu_0 N^2\; h}{2\pi}\;\ln\frac{b}{a} \qquad \text{-----------------} \quad \textbf{(1)}$$

The self inductance of a toroid is:

$$L = \frac{\phi_B}{I} \qquad \text{-----------------} \quad \textbf{(2)}$$

$$\because \phi_B \propto I$$

$$\because \phi_B = L\; I$$

By comparing (1) and (2), we get:

$$L = \frac{\mu_0 N^2\, h}{2\pi}\ \ln\frac{b}{a}$$

So, the inductance of a toroid depends only on geometrical factors.

Inductors with Magnetic Materials

When a magnetic field B_0 acts on a ferromagnetic material, then total magnetic field B is increased which can be expressed as:

$$B = \mu_r\, B_0$$

Where μ_r is the relative permeability of permeability constant of the magnetic material.

Similarly, the inductance of a coil is increased, if some magnetic material is placed inside the coil. If L_0 and L is the inductance with and without the magnetic material, respectively, then:

$$L = \mu_r\, L_0 \qquad \text{------------------} \quad \textbf{(1)}$$

Now the inductance of solenoid without magnetic material is:

$$L_0 = \mu_0 n^2 l\, A$$

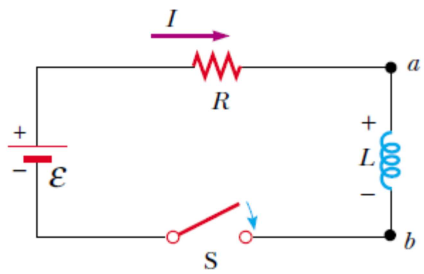
Putting values of L_0 in equation (1), we have:

$$L = \mu_r\, \mu_0 n^2 l\, A$$

The effective values of permeability constant μ_r for a ferromagnetic material have the values in range of 10^3 to 10^4 . Therefore the inductance of a coil can be increased by a factor of 10^3 to 10^4 with the core of a ferromagnetic substance.

Growth of Current in LR Circuit

Consider a series circuit of a resistance R and inductance connected in series with a battery of emf ε as shown in the figure.



When the switch is open, no current will flow through the circuit. So at time $t = 0, I = 0$.

When the switch is closed, the current starts flowing through the circuit. But due to inductance, this current takes some time to get its maximum value. (During this period, the current is called transient current)

Initially, when the current is changing, the back emf $L\,\frac{dI}{dt}$ appears across the inductance coil which dies away with the passage of time. And its value falls to zero, when the current is maximum.

Let V_R and V_L are the values of potential drop across resistance R and inductance L, respectively. Then, by applying the Kirchhoff's rule:

$$\varepsilon - V_R - V_L = 0$$

$$V_L = \varepsilon - V_R$$

$$L\,\frac{dI}{dt} = \varepsilon - IR$$

$$\frac{dI}{\varepsilon - IR} = \frac{dt}{L}$$

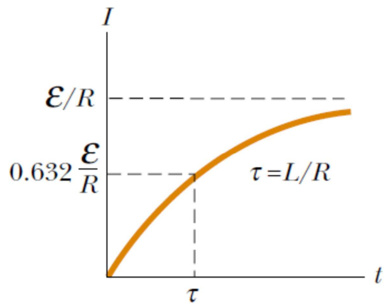
Multiplying both sides by $-R$, we get:

$$\frac{-RdI}{\varepsilon - IR} = -R\,\frac{dt}{L}$$

Integrating between the limits,

At $t = 0, I = 0$;

At $t = t, I = I$



$$\int_0^I \frac{-RdI}{\varepsilon - IR} = \int_0^t -R\frac{dt}{L}$$

$$|\ln(\varepsilon - IR)|_0^I = -\frac{R}{L}\ |t|_0^t$$

$$\ln(\varepsilon - IR) - \ln(\varepsilon) = -\frac{R}{L}\ t$$

$$\ln\left(\frac{\varepsilon - IR}{\varepsilon}\right) = -\frac{R}{L}\ t$$

$$\frac{\varepsilon - IR}{\varepsilon} = e^{\left(-\frac{R}{L}t\right)}$$

$$IR = \varepsilon - \varepsilon\, e^{\left(-\frac{R}{L}t\right)}$$

$$IR = \varepsilon\left(1 - e^{\left(-\frac{R}{L}t\right)}\right)$$

$$I = \frac{\varepsilon}{R}\left(1 - e^{\left(-\frac{t}{L/R}\right)}\right)$$

Let $\tau_L = \frac{L}{R}$ = Inductive time constant and $\frac{\varepsilon}{R} = I_0$ = Maximum Current

$$I = I_0\left(1 - e^{\left(-\frac{t}{\tau_L}\right)}\right)$$

This is the equation of growth of current

Special Cases:

At time $t = \tau_L$

$$I = I_0\left(1 - e^{\left(-\frac{\tau_L}{\tau_L}\right)}\right) = I_0(1 - e^{-1}) = I_0(1 - 0.37)$$

$$I = 0.63\ I_0$$

At $t = \infty$

$$I = I_0\left(1 - e^{(-\infty)}\right) = I_0\left(1 - \frac{1}{\infty}\right) = I_0(1 - 0)$$

$$I = I_0$$

Inductive Time Constant

"It is the interval of time during which the current grows to 63 % *of its maximum value"*.

The graph shows that current through the circuit increases exponentially.

As

$$I = I_0\left(1 - e^{\left(-\frac{R}{L}t\right)}\right) \quad \text{------------------} \quad \textbf{(1)}$$

$$V_R = IR = I_0R\left(1 - e^{\left(-\frac{R}{L}t\right)}\right) = \varepsilon\left(1 - e^{\left(-\frac{R}{L}t\right)}\right) \quad \text{------------------} \quad \textbf{(2)}$$

i.e., V_R increases exponentially upto ε.

Differentiating equation (1) with respect to time, we get:

$$\frac{dI}{dt} = -\frac{R}{L}I_0\left(0 - e^{\left(-\frac{R}{L}t\right)}\right)$$

$$\Rightarrow V_L = L\frac{dI}{dt} = I_0R\ e^{\left(-\frac{R}{L}t\right)} = \varepsilon\ e^{\left(-\frac{R}{L}t\right)} \quad \text{------------------} \quad \textbf{(3)}$$

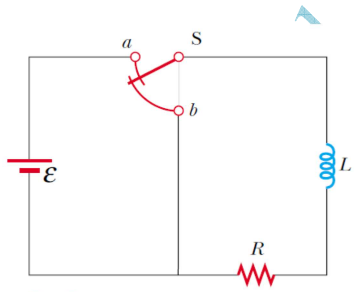
i.e., V_L decreases exponentially upto 0.

Decay of Current in LR Circuit

When the S is connected to the b, as shown in the figure, the current start decaying exponentially with time and becomes zero after some time.

If V_R and V_L are the values of potential difference across the resistance R and inductance L, respectively. Then by applying the Kirchhoff's Rule:

$$V_R + V_L = 0$$
$$IR + L\frac{dI}{dt} = 0$$
$$L\frac{dI}{dt} = -IR$$
$$\frac{dI}{I} = -\frac{R}{L}\,dt$$



Integrating between the limits,

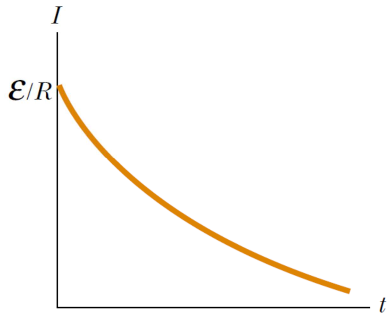
At $t = 0, I = I_0$;
At $t = t, I = I$

$$\int_{I_0}^{I}\frac{dI}{I} = \int_0^t -R\frac{dt}{L}$$
$$|\ln(I)|_{I_0}^{I} = -\frac{R}{L}\,|t|_0^t$$
$$\ln(I) - \ln(I_0) = -\frac{R}{L}\,t$$
$$\ln\left(\frac{I}{I_0}\right) = -\frac{R}{L}\,t$$
$$\frac{I}{I_0} = e^{\left(-\frac{R}{L}t\right)}$$
$$I = I_0\,e^{\left(-\frac{R}{L}t\right)}$$
$$I = I_0\,e^{\left(-\frac{t}{L/R}\right)}$$

Let $\tau_L = \frac{L}{R}$ = Inductive time constant

$$I = I_0\,e^{\left(-\frac{t}{\tau_L}\right)}$$

This is the equation of decay of current in an LR circuit.



Special Cases:

At time $t = \tau_L$

$$I = I_0\,e^{\left(-\frac{t}{\tau_L}\right)} = I_0(\,e^{-1}) = I_0(\,0.37)$$
$$I = 0.37\ I_0$$

At $t = \infty$

$$I = I_0\left(\,e^{(-\infty)}\right) = I_0\left(\frac{1}{\infty}\right) = I_0(\,0)$$
$$I = 0$$

Inductive Time Constant

"*It is the interval of time during which the current decays to* 63 % *of its maximum value*".

As $\;I = I_0\,e^{\left(-\frac{t}{\tau_L}\right)}$ ------------------ **(1)**

The potential difference across R is

$$V_R = IR = I_0R\ e^{\left(-\frac{R}{L}t\right)}$$

$V_R = \varepsilon\ \ e^{\left(-\frac{R}{L}t\right)}$ ------------------ **(2)**

Differentiating equation (1) with respect to time, we get:

$$\frac{dI}{dt} = -\frac{R}{L}I_0\ e^{\left(-\frac{R}{L}t\right)}$$

$$\Rightarrow V_L = L\frac{dI}{dt} = -I_0 R \ e^{\left(-\frac{R}{L}t\right)}$$

$$\Rightarrow V_L = -\varepsilon \ e^{\left(-\frac{R}{L}t\right)} \qquad \text{-----------------} \quad \mathbf{(3)}$$

Energy Stored in the Magnetic Field

Consider a resistor and an inductor is connected in series with a battery of emf ε. According to the Kirchhoff's 2nd rule;

$$\varepsilon - V_R - V_L = 0$$

$$\varepsilon = V_R + V_L$$

$$\varepsilon = IR + L\,\frac{dI}{dt}$$

Multiplying both sides by 'I', we have:

$$\varepsilon I = I^2 R + LI\,\frac{dI}{dt}$$

Here εI is the power supplied by the source, I^2R is the power dissipated in resistance and $LI\,\frac{dI}{dt}$ is the energy supplied per time in the inductance coil, where the magnetic field also exists.

If $U_B$is the total energy stored in the magnetic field of inductance coil, then the energy stored per unit time is:

$$\frac{dU_B}{dt} = LI\,\frac{dI}{dt}$$

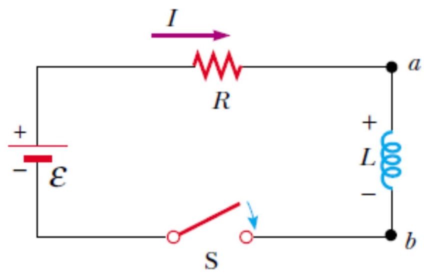
$$dU_B = LI\,dI$$

Integrating both sides, we get:

$$\int_0^{U_B} dU_B = \int_0^{I} LI\,dI$$

$$U_B = \frac{1}{2}LI^2$$

This is the expression of energy stored in the magnetic field of a current carrying inductor. This equation is similar to the expression of energy stored in the electric field of a capacitor.



$$U_E = \frac{1}{2}\,\frac{q^2}{C}$$

The energy stored in the inductor can be dissipated through joule heating in the resistor, in the similar way the energy stored in the capacitor is dissipated through the joule heating in the resistor during the discharging of a capacitor.

Energy Density

The energy density u_B in case of an inductor can be find out by using expression:

$$Energy\ Density = \frac{Energy\ Stored}{Volume}$$

If U_B is the energy stored in the magnetic field of an inductor of length l having cross-sectional area A, then the energy density in the expression will be:

$$u_B = \frac{U_B}{A\,l}$$

As $U_B = \frac{1}{2}LI^2$, therefore

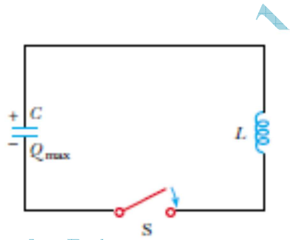
$$\Rightarrow u_B = \frac{\left(\frac{1}{2}LI^2\right)}{A\,l} = \frac{LI^2}{2A\,l}$$

In case of a current carrying coil, the induce L can be find out by using expression: $L = \mu_0 n^2 l\,A$

$$\Rightarrow u_B = \frac{(\mu_0 n^2 l\,A)I^2}{2A\,l} = \frac{\mu_0 n^2 l\,AI^2}{2A\,l}$$

$$\Rightarrow u_B = \frac{\mu_0 n^2 I^2}{2} = \frac{{\mu_0}^2 n^2 I^2}{2\mu_0}$$

The magnetic field strength inside a current carrying coil is $B = \mu_0 n\,I$



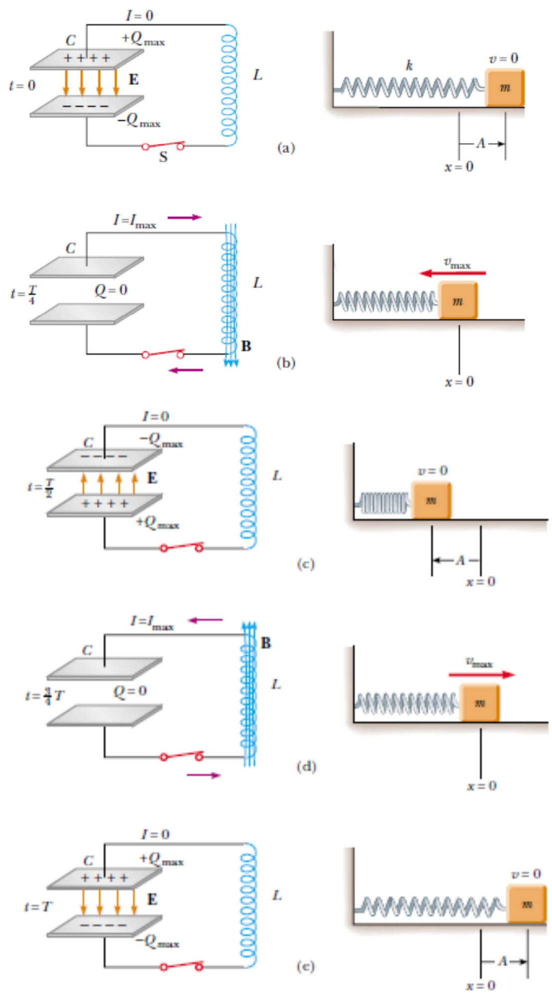
$$\Rightarrow u_B = \frac{B^2}{2\mu_0}$$

This expression is similar to the energy density inside the electric field of a capacitor, which is expressed as:

$$u_E = \frac{\varepsilon_0\,E^2}{2}$$

where ε_0 is the permittivity of free space and E is the electric field strength inside the capacitor.

Electromagnetic Oscillations (Qualitative discussion)



Consider an oscillating circuit which consists of a capacitor and an inductor. If the capacitor is initially charged and the switch is then closed, we find that both the current in the circuit and the charge on the capacitor oscillate between maximum positive and negative values. If the resistance of the circuit is zero, no energy is transformed to internal energy. In the following analysis, we neglect the resistance in the circuit. We also assume an idealized situation in which energy is not radiated away from the circuit.

When the capacitor is fully charged, the energy U_E in the circuit is stored in the electric field of the capacitor and is equal to

$$U_E = \frac{1}{2}\,\frac{(q_{max})^2}{C}$$

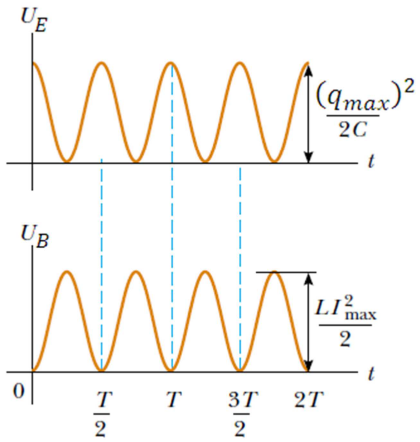
At this time, the current in the circuit is zero, and therefore no energy is stored in the inductor.

$$U_B = 0$$

As the switch S is closed, the capacitor begins to discharge and the energy stored in its electric field decreases. The discharge of the capacitor represents a current in the circuit, and hence some energy is now stored in the magnetic field of the inductor. Thus, energy is transferred from the electric field of the capacitor to the magnetic field of the inductor. When the capacitor is fully discharged, it stores no energy. At this time, the current reaches its maximum value, and all of the energy is stored in the inductor.

$$(U_E)_{min} = 0,\ (U_B)_{max} = \frac{1}{2}L(I_{max})^2$$

The current continues in the same direction, decreasing in magnitude, with the capacitor eventually becoming fully charged again but with the polarity of its plates now opposite the initial polarity. This is followed by another discharge until the circuit returns to its original state of maximum charge q_{max}. The energy continues to oscillate between inductor and capacitor. When the energy stored in the capacitor is maximum, the energy stored in inductor is minimum and vice versa. But the total energy $U = U_E + U_B$ remains constant.



As the energy in such circuit is used to oscillate between capacitor and inductor, therefore, it is also called 'Oscillating Circuit'.

LC Circuit: Electromagnetic Oscillations (Quantitative analysis)

Let U_E and U_B are the values of energy stored in the electric field of capacitor and magnetic field of an inductor, respectively.

So the total energy U stored in the circuit is:

$$U = U_E + U_B$$

$$U = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}LI^2$$

Differentiating the above equation, we get

$$\frac{dU}{dt} = \frac{d}{dt}\left[\frac{1}{2}\frac{q^2}{C} + \frac{1}{2}LI^2\right]$$

$$\because U = cosntant$$

$$0 = \frac{q}{C}\frac{dq}{dt} + LI\frac{dI}{dt}$$

$$\Rightarrow \frac{q}{C}\frac{dq}{dt} + LI\frac{dI}{dt} = 0$$

$$\because I = \frac{dq}{dt}$$

$$\Rightarrow \frac{q}{C}\frac{dq}{dt} + L.\frac{dq}{dt}.\frac{d}{dt}\left(\frac{dq}{dt}\right) = 0$$

$$\Rightarrow \frac{dq}{dt}\left[\frac{q}{C} + L.\frac{d}{dt}\left(\frac{dq}{dt}\right)\right] = 0$$

$$\Rightarrow \frac{q}{C} + L\frac{d^2q}{dt^2} = 0$$

$$\Rightarrow \frac{d^2q}{dt^2} + \frac{q}{LC} = 0 \qquad \text{-------------- } \mathbf{(1)}$$

For a block spring system, the differential equation of a mass spring system is:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

The solution of this equation is:

$$x = x_m \cos(wt + \Phi)$$

Where x_m is the amplitude of oscillating mass spring system and Φ is its initial phase. Similarly, the solution of equation (1) will be:

$$q = q_m \cos(wt + \Phi)$$

And

$$\frac{dq}{dt} = -\, w\, q_m \sin(wt + \Phi)$$

$$\frac{d^2q}{dt^2} = -\, w^2\, q_m \cos(wt + \Phi)$$

Putting the values of q and $\frac{d^2q}{dt^2}$ in equation (1), we get:

$$-\, q_m \cos(wt + \Phi)\left[w^2 - \frac{1}{LC}\right] = 0$$

$$\Rightarrow w^2 - \frac{1}{LC} = 0$$

$$\Rightarrow w = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow f = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}$$

This is the expression of frequency of the LC circuit of electromagnetic oscillations.

The energy stored in the electric field of a capacitor is:

$$U_E = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\,[q_m \cos(wt + \Phi)]^2$$

$$U_E = \frac{1}{2C}\,[q_m^2 \cos^2(wt + \Phi)]$$

The energy stored in the magnetic field of an inductor will be:

$$U_B = \frac{1}{2}LI^2$$

$$\Rightarrow U_B = \frac{1}{2}L\left[\frac{dq}{dt}\right]^2 = \frac{1}{2}L\left[\frac{d}{dt}\{q_m \cos(wt + \Phi)\}\right]^2$$

$$\Rightarrow U_B = \frac{1}{2}L[w^2 q_m^2 \sin^2(wt + \Phi)]$$

$$\Rightarrow U_B = \frac{1}{2}L\left[\frac{1}{LC}\, q_m^2 \sin^2(wt + \Phi)\right]$$

$$\Rightarrow U_B = \frac{1}{2C}\, q_m^2 \sin^2(wt + \Phi)$$

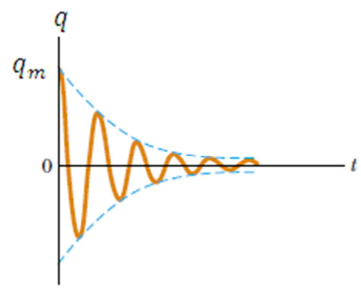
The total energy stored in the LC circuit is:

$$U = U_E + U_B = \frac{1}{2}\, q_m^2 \cos^2(wt + \Phi) + \frac{1}{2C}\, q_m^2 \sin^2(wt + \Phi)$$

$$U = \frac{1}{2}\, q_m^2[\cos^2(wt + \Phi) + \sin^2(wt + \Phi)]$$

$$U = \frac{1}{2}\, q_m^2$$

As the q_m is the maximum value of charged stored in the capacitor, so the total energy stored in LC circuit is constant.



Damped and Forced Oscillations

A resistance is always present in every real LC circuit. When we take this resistance into account, we find that the total electromagnetic energy is not constant but decreases with time as it is dissipated as internal energy in the resistor. For a LC circuit, the total energy U of the circuit is described as:

$$U = U_E + U_B$$

Where U_E is the energy stored in the electric field of a charged capacitor and U_B is the energy stored in the magnetic field of inductor.

$$U = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}LI^2$$

Differentiating the above equation, we get

$$\frac{dU}{dt} = \frac{d}{dt}\left[\frac{1}{2}\frac{q^2}{C} + \frac{1}{2}LI^2\right] \qquad \text{------------------- } \mathbf{(1)}$$

The total energy of LC circuit is no longer constant but rather

$$\frac{dU}{dt} = -I^2R$$

Thus the equation (1) will become:

$$\frac{d}{dt}\left[\frac{1}{2}\frac{q^2}{C} + \frac{1}{2}LI^2\right] = -I^2R$$

$$\Rightarrow \frac{q}{C}\frac{dq}{dt} + LI\frac{dI}{dt} = -I^2R$$

$$\because I = \frac{dq}{dt}$$

$$\Rightarrow \frac{q}{C}\frac{dq}{dt} + L.\frac{dq}{dt}.\frac{d}{dt}\left(\frac{dq}{dt}\right) = -\left(\frac{dq}{dt}\right)^2 R$$

$$\Rightarrow \frac{dq}{dt}\left[\frac{q}{C} + L.\frac{d}{dt}\left(\frac{dq}{dt}\right)\right] = -\left(\frac{dq}{dt}\right)^2 R$$

$$\Rightarrow \frac{q}{C} + L.\frac{d}{dt}\left(\frac{dq}{dt}\right) = -\frac{dq}{dt}R$$

$$\Rightarrow L\frac{d^2q}{dt^2} + \frac{dq}{dt}R + \frac{q}{C} = 0$$

The general solution of the above equation will be:

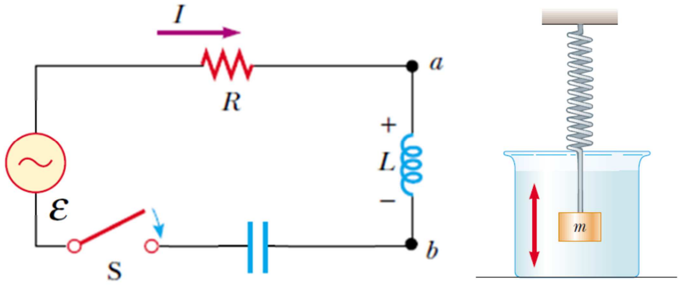
$$q = q_m e^{\left(-\frac{Rt}{2L}\right)}\cos(\omega' t + \Phi)$$

Here $\omega' = \sqrt{\omega^2 - \left(\frac{R}{2L}\right)^2}$

The figure shows the charged stored on the capacitor in a damped LC circuit as the function of time. The current decreases exponentially with time. Moreover the frequency ω' is strictly less than the frequency ω of undammed oscillations.

Forced Oscillations and Resonance

The RLC series circuit is similar to a mechanical system consisting of a simple harmonic oscillator in a damping medium. The oscillating system consist of mass m attached to a spring of spring constant k.



If the frequency of the oscillator is equal to the natural frequency of system $\frac{k}{m}$, the there will be resonance and the amplitude of the oscillation will be maximum.

In the similar way, if ω' is the frequency of sinusoidal voltage source, then the sinusoidal voltage is expressed as:

$$\varepsilon = \varepsilon_m \cos \omega' t$$

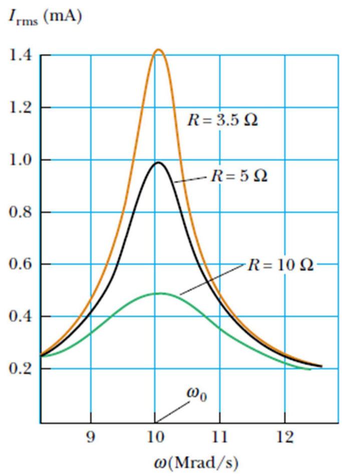
Where ε_m is the peak value of sinusoidal voltage. The current flowing through the circuit will be:

$$I = I_m \cos(\omega' t - \Phi)$$

Where I_m is the maximum value of sinusoidal current.

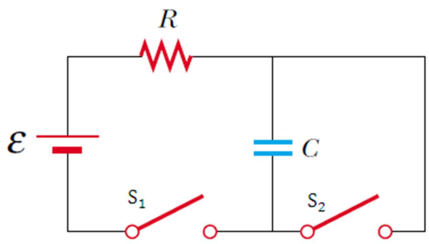
If the angular frequency of external voltage source is equal to the natural frequency $\omega = \frac{1}{\sqrt{LC}}$ of the circuit, then there will be resonance and the maximum current will flow through the circuit.

The resistance R provides the damping in the circuit. Greater the resistance, smaller will the current flow through the circuit. Such oscillations which are produced in oscillating system are called forced oscillations.



Growth of Current in RC Series Circuit

Consider a resistance R and a capacitance C is connected in series with a battery of emf ε as shown in the figure. The S_1 and S_2 are the two switches.



When the switch S_1 is closed, keeping the switch S_2, the capacitor begins to charge and the voltage across the capacitor V_C begins to increase. Let V_R is the potential drop across the resistor, then by applying the Kirchhoff's law:

$$\varepsilon - V_R - V_C = 0$$

$$\varepsilon - IR - \frac{q}{C} = 0$$

$$\varepsilon - \frac{q}{C} = IR$$

$$\because I = \frac{dq}{dt}$$

$$\varepsilon - \frac{q}{C} = R\,\frac{dq}{dt}$$

$$\frac{1}{R}dt = \frac{dq}{\left(\varepsilon - \frac{q}{C}\right)}$$

Multiplying equation by '$-\frac{1}{C}$' on both sides:

$$\frac{-\frac{1}{C}dq}{\left(\varepsilon - \frac{q}{C}\right)} = -\frac{1}{RC}dt$$

Integrating the above equation, we get:

$$\int_0^q \frac{-\frac{1}{C}dq}{\left(\varepsilon - \frac{q}{C}\right)} = -\frac{1}{RC}\int_0^t dt$$

$$\Rightarrow \left|\ln\left(\varepsilon - \frac{q}{C}\right)\right|_0^q = -\frac{1}{RC}\ |t|_0^t$$

$$\Rightarrow \ln\left(\varepsilon - \frac{q}{C}\right) - \ln(\varepsilon) = -\frac{1}{RC}\ (t - 0)$$

$$\Rightarrow \ln\left(\frac{\varepsilon - \frac{q}{C}}{\varepsilon}\right) = -\frac{t}{RC}$$

$$\Rightarrow \ln\left(1 - \frac{q}{\varepsilon C}\right) = -\frac{t}{RC}$$

$$\Rightarrow 1 - \frac{q}{\varepsilon C} = e^{\left(-\frac{t}{RC}\right)}$$

$$\Rightarrow \frac{q}{\varepsilon C} = 1 - e^{\left(-\frac{t}{RC}\right)}$$

$$\Rightarrow q = \varepsilon C\left(1 - e^{\left(-\frac{t}{RC}\right)}\right)$$

Where $\tau_c = RC$ is the capacitive time constant, therefore:

$$\Rightarrow q = \varepsilon C\left(1 - e^{\left(-\frac{t}{\tau_c}\right)}\right) \qquad \text{-------------------} \ \mathbf{(1)}$$

When $t = \infty$, the capacitor will fully charged to its saturation value q_0

$$q_0 = \varepsilon C\left(1 - e^{\left(-\frac{\infty}{\tau_c}\right)}\right) = \varepsilon C\left(1 - e^{(-\infty)}\right) = \varepsilon C\left(1 - \frac{1}{e^{\infty}}\right) = \varepsilon C\left(1 - \frac{1}{\infty}\right) = \varepsilon C$$

The equation (1) will become:

$$q = q_0\left(1 - e^{\left(-\frac{t}{\tau_c}\right)}\right) \qquad \text{-------------------} \ \mathbf{(2)}$$

Growth of Current

Differentiating equation (2) with respect to time, we get:

$$\frac{dq}{dt} = \frac{d}{dt}\left[q_0\left(1 - e^{\left(-\frac{t}{\tau_c}\right)}\right)\right] = \frac{d}{dt}\left[q_0 - q_0\, e^{\left(-\frac{t}{RC}\right)}\right]$$

$$I = 0 - q_0\left(-\frac{1}{RC}\right)\ e^{\left(-\frac{t}{RC}\right)}$$

$$I = \frac{1}{R}\frac{q_0}{C}\ e^{\left(-\frac{t}{RC}\right)}$$

$$\because \frac{q_0}{C} = \varepsilon$$

$$I = \frac{\varepsilon}{R}\ e^{\left(-\frac{t}{RC}\right)}$$

$$\because \frac{\varepsilon}{R} = I_0$$

$$I = I_0\ e^{\left(-\frac{t}{RC}\right)}$$

Special Cases

At $t = 0$

$$q = \varepsilon C\left(1 - e^{\left(-\frac{0}{\tau_c}\right)}\right) = \varepsilon C(1 - e^{0}) = \varepsilon C(1 - 1)$$

$$q = 0$$

At $t = \tau_c$

$$q = \varepsilon C\left(1 - e^{\left(-\frac{\tau_c}{\tau_c}\right)}\right) = \varepsilon C(1 - e^{(-1)}) = \varepsilon C\left(1 - \frac{1}{e}\right)$$

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