GAUSS'S LAW

29.1 Electric Flux

The number of electric lines of force passing normally through a certain area is called the electric flux. It is measured by the product of area and the component of electric field intensity normal to the area. It is denoted by the symbol Φ_e.

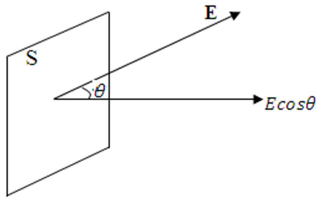
Consider a surface 'S' placed in a uniform electric field of intensity 'E'. Let 'A' be the vector area of the surface. The component of E perpendicular to the area A is $E\cos\theta$ as shown in the figure below.

The electric flux through the surface S is given by

$$\Phi_e = A\,(E\cos\theta)$$

$$\Phi_e = EA\cos\theta$$

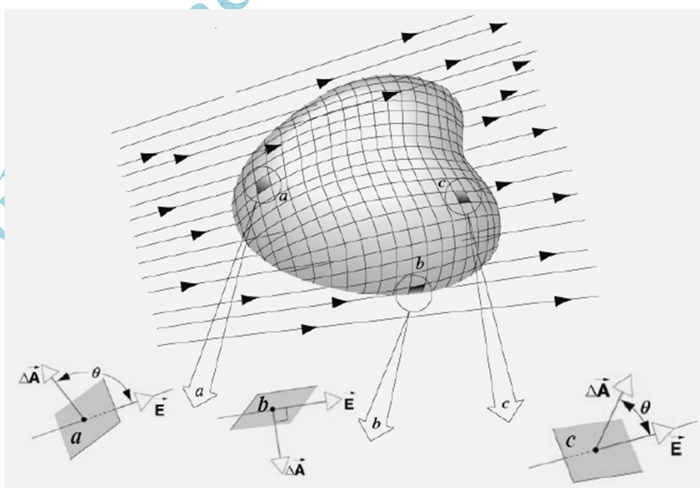
$$\Phi_e = \mathbf{E}.\mathbf{A} \qquad \text{----------------------} \qquad \mathbf{(1)}$$



Thus the electric flux is the scalar product of electric field intensity and the vector area. The SI unit of the electric flux is $\frac{Nm^2}{C}$.

If the electric field is not uniform, then we divide the surface into number of small patches each of area $\Delta\mathbf{A}$. Then equation (1) becomes,

$$\Phi_e = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{E_i}.\Delta\mathbf{A_i}$$



When $n \to \infty$, or $\Delta A \to 0$, then the sigma is replaced by the surface integral i.e.,

$$\Phi_e = \int_S \mathbf{E}.\mathbf{dA} \qquad \text{----------------------} \qquad \mathbf{(2)}$$

By convention, the outward flux is taken as positive and inward flux is taken as negative.

29.2 Gauss's Law

Statement

The total electric flux through any close surface is $\frac{1}{\varepsilon_0}$ times the total charge enclosed by the surface.

Explanation

The Gauss's law gives the relation between total flux and total charge enclosed by the surface. Consider a collection of positive and negative charges in a certain region of space.

According to Gauss's law,

$$\Phi_e = \frac{q}{\varepsilon_0}$$

$$\varepsilon_0 \Phi_e = q$$

where q is the net charge enclosed by the surface.

But $\Phi_e = \oint \mathbf{E.dA}$

$$\varepsilon_0 \oint \mathbf{E.dA} = q$$

$$\oint \mathbf{E.dA} = \frac{q}{\varepsilon_0}$$

29.3 Differential Form of Gauss's Law

If the charge is distributed into a volume having uniform volume charge density 'ρ', then according to Gauss's law,

$$\Phi_e = \frac{1}{\varepsilon_0}\int_{\mathbf{v}} \rho\, dv$$

$$\therefore \Phi_e = \oint \mathbf{E.dA}$$

$$\oint \mathbf{E.dA} = \frac{1}{\varepsilon_0}\int_{\mathbf{v}} \rho\, dv \qquad \text{----------------------(3)}$$

We know that by Divergence theorem,

$$\oint \mathbf{E.dA} == \int_{\mathbf{v}} div\, \mathbf{E}\, dv$$

Eq (3)becomes

$$\int_{\mathbf{v}} div\, \mathbf{E}\, dv = \frac{1}{\varepsilon_0}\int_{\mathbf{v}} \rho\, dv$$

$$\int_{\mathbf{v}} div\, \mathbf{E}\, dv - \frac{1}{\varepsilon_0}\int_{\mathbf{v}} \rho\, dv = \mathbf{0}$$

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$$\int_{\mathbf{v}}(div\,\mathbf{E} - \frac{1}{\varepsilon_0}\rho\,)\,dv = \mathbf{0}$$

As $dv \neq \mathbf{0}$

$$div\,\mathbf{E} - \frac{1}{\varepsilon_0}\rho = 0$$

$$div\,\mathbf{E} = \frac{1}{\varepsilon_0}\rho$$

This is differential form of Gauss's law.

29.4 Integral Form of Gauss's Law

According to Gauss's law

$$\Phi_e = \oint \mathbf{E.dA} = \frac{q}{\varepsilon_0}$$

where 'q' is the total charge enclosed.

If the charge is uniformly distributed into a volume having charge density 'ρ', then

$$\Phi_e = \oint \mathbf{E.dA} = \frac{1}{\varepsilon_0}\int_{\mathbf{v}} \rho\, dv \qquad \text{---------------------- } \mathbf{(4)}$$

If the charge is uniformly distributed over a surface having a surface charge density 'σ', then

$$\Phi_e = \oint \mathbf{E.dA} = \frac{1}{\varepsilon_0}\int_{\mathbf{s}} \sigma\, dA \qquad \text{---------------------- } \mathbf{(5)}$$

Equation (4) and (5) are the integral form of Gauss's law.

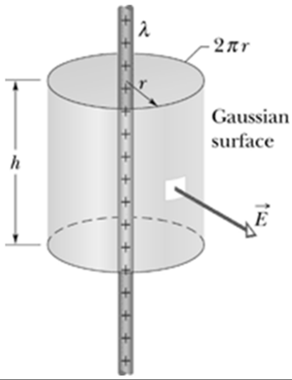
29.5 Applications of Gauss's law

Gauss's law can be used to calculate the electric field intensity due to certain charge distributions if the charge distribution has the greater symmetry.

29.5.1 Electric Field due to Infinite Line of Charge

Consider a section of infinite line of charge having uniform linear charge density 'λ' as shown in the figure below.

We want to find out electric field intensity at any point 'P' which is at distance 'r' from the wire. For this we consider cylindrical Gaussian surface which passes through point 'P'. The electric flux passing through the cylinder is given as



$$\Phi_e = \oint_{\mathbf{S}} \mathbf{E.ds}$$

The surface 'S' of the cylinder consist of three parts i.e., S_1, S_2 and S_3, where

S_1 = Area of top cross section of cylindrical Gaussian surface

S_2 = Area of bottom cross section of cylindrical Gaussian surface

S_3 = Area of curved part of Gaussian surface

Thus

$$\Phi_e = \int_{S_1} \mathbf{E.ds} + \int_{S_2} \mathbf{E.ds} + \int_{S_3} \mathbf{E.ds}$$

Now

$$\int_{S_1} \mathbf{E.ds} = \int_{S_2} \mathbf{E.ds} = 0 \text{ because } \theta = 90^\circ$$

And $$\int_{S_3} \mathbf{E.ds} = \int_{S_3} E\, ds \text{ because } \theta = 0^\circ$$

Therefore

$$\Phi_e = \int_{S_3} E\, ds = E \int_{S_3} ds \qquad \therefore E \text{ is constant}$$

For cylindrical symmetry $\int_{S_3} ds = 2\pi rh$

$$\Phi_e = E(2\pi rh) = 2\pi rh\, E \qquad \text{---------------------- (6)}$$

By Gauss's law

$$\Phi_e = \frac{q}{\varepsilon_0} = \frac{\lambda\, h}{\varepsilon_0} \qquad \text{---------------------- (7)}$$

$$\therefore \lambda = \frac{q}{h}$$

Comparing Eq. (6) and (7), we get

$$2\pi rh\, E = \frac{\lambda\, h}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi r\varepsilon_0}$$

If '$\hat{r}$' gives the direction of electric field intensity, then

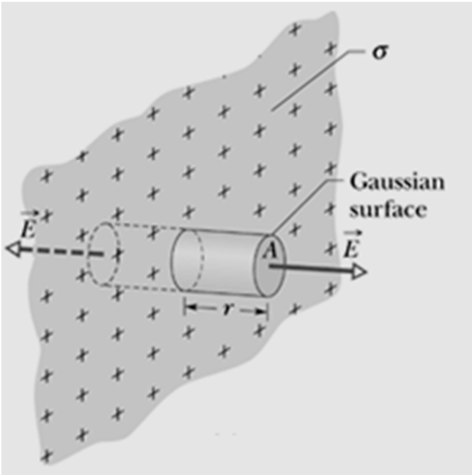
$$\mathbf{E} = \frac{\lambda}{2\pi r\varepsilon_0}\hat{r} \qquad \text{---------------------- (8)}$$

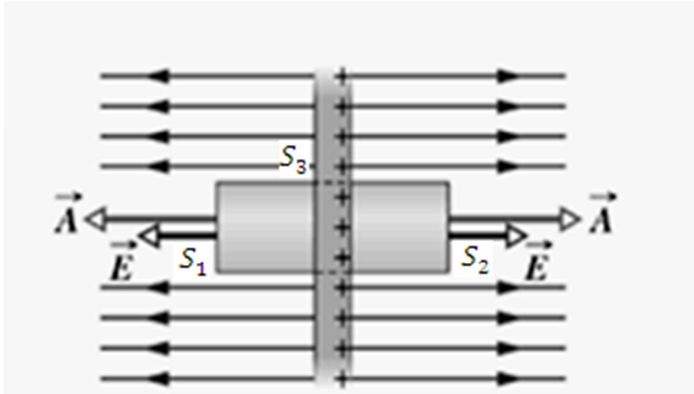
This expression gives the electric field intensity due to infinite line of charge.

29.5.2 Electric Field at a Point Due to Infinite Sheet of Charge

Consider an infinite sheet of charge having constant surface charge density 'σ'. The figure shows a small portion of such sheet.

We want to find electric field intensity at point 'P' which is at the distance 'r' from sheet. For this we consider a cylindrical Gaussian surface as shown in the figure below.





The net electric flux passing through the cylinder is given as

$$\Phi_e = \int_S \mathbf{E.dA}$$

We divide the cylindrical Gaussian surface into three parts i.e., S_1, S_2 and S_3, where

S_1 = Left cross sectional area of cylindrical Gaussian surface

S_2 = Right cross sectional area of cylindrical Gaussian surface

S_3 = Area of curved of cylindrical Gaussian surface

Thus

$$\Phi_e = \int_{S_1} \mathbf{E}.d\mathbf{A_1} + \int_{S_2} \mathbf{E}.d\mathbf{A_2} + \int_{S_3} \mathbf{E}.d\mathbf{A_3}$$

Now $\int_{S_3} \mathbf{E}.d\mathbf{A_3} = 0$ because $\theta = 90^\circ$

Therefore

$$\Phi_e = \int_{S_1} \mathbf{E}.d\mathbf{A_1} + \int_{S_2} \mathbf{E}.d\mathbf{A_2}$$

In case of surfaces S_1 and S_2, **E** & **dA** are parallel to each other i.e., $\theta = 0^\circ$ and $dA_1 = dA_2 = dA$.

$$\Phi_e = \int_{S_1} E\,dA + \int_{S_2} E\,dA$$

$\therefore E \text{ is constant}$

$$\Phi_e = E\int_{S_1} dA + E\int_{S_2} dA$$

$$\Phi_e = E\,A + E\,A = 2\,E\,A \qquad \text{---------------------- (9)}$$

According to Gauss's law

$$\Phi_e = \oint \mathbf{E.dA} = \frac{\sigma A}{\varepsilon_0} \qquad \text{---------------------- (10)}$$

$$\therefore \sigma = \frac{q}{A}$$

Comparing Eq. (9) and (10), we get

$$2\ E\ A = \frac{\sigma A}{\varepsilon_0}$$

$$E = \frac{\sigma}{2\varepsilon_0}$$

If '$\hat{r}$' gives the direction of electric field intensity, then

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\ \hat{r}$$

This is the expression of electric field intensity due to infinite sheet of charge.

29.5.3 Electric Field due to Spherical Shell of Charge

Consider a thin spherical shell of radius 'R' which have the charge 'q' with constant surface charge density 'σ'. The surface charge density

$$\sigma = \frac{q}{A}$$

$$\sigma = \frac{q}{4\pi r^2}$$

$$\therefore A = 4\pi r^2\ (Surface\ area\ of\ sphere)$$

$$q = 4\pi r^2\ \sigma \qquad \text{----------------------- (11)}$$

Theorem 1: A uniform spherical shell of charge behaves, for all external points, as if all its charge were concentrated at its center.

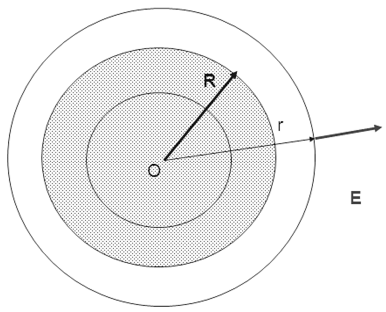
Proof:

Consider a point 'P' outside the shell. We want to find out electric field intensity due to this charge distribution. For this we consider a spherical Gaussian surface of radius $r > R$ which passes through point 'P' as shown in the figure below.

According to Gauss's law,

$$\oint \mathbf{E}.\mathbf{dA} = \frac{q}{\varepsilon_0}$$

$$\oint E\ dA = \frac{q}{\varepsilon_0} \qquad \therefore$$



$$Angle\ between\ E\ and\ dA\ is\ zero$$

$$E \oint dA = \frac{q}{\varepsilon_0}$$

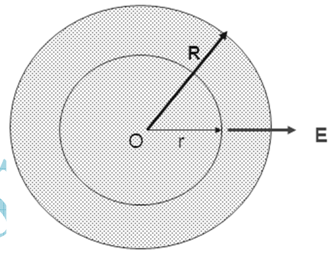
$$E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0}.\frac{q}{r^2}$$

Thus the uniform spherical shell of charge behaves like a point charge for all the points outside the shell.

Theorem 2: A uniform spherical shell of charge exerts no electrostatic force on a charged particle placed inside the shell.

Consider a point 'P' inside the shell. We want to fine out electric field intensity 'E' at point 'P' due to this symmetrical charge distribution. For this we consider a spherical Gaussian surface of radius $r < R$ which passes through point 'P' as shown in the figure below.



According to Gauss's law,

$$\oint \mathbf{E}.\mathbf{dA} = \frac{q}{\varepsilon_0}$$

Because the Gaussian surface enclose no charge, therefore 'q = 0',

$$\oint \mathbf{E}.\mathbf{dA} = 0$$

$$\oint E\, dA = 0 \qquad \therefore Angle\ between\ E\ and\ dA\ is\ zero$$

$$E \oint dA = 0 \qquad \therefore E\ is\ constant$$

As $dA \neq 0$, therefore

$$E = 0$$

So the electric field does not exist inside a uniform shell of charge. So the test charge placed inside the charged shell would experience no force.

29.5.4 Electric Field due to Spherical Charge

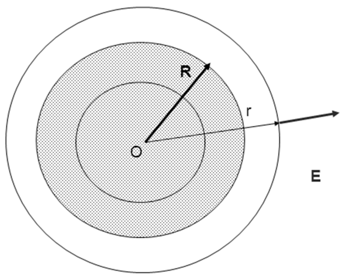
Case 1: When the point 'P' is outside the sphere of charge

Consider a spherical distribution of charge of radius 'R' with the uniform volume charge density 'ρ'. We want to find the electric field at point 'P' at a distance $r > R$ from the center of charged sphere. For we consider a spherical Gaussian surface which passes through point 'P' as shown in the figure below.

According to Gauss's law,

$$\oint \mathbf{E}.\mathbf{dA} = \frac{q}{\varepsilon_0}$$

$$\oint E\, dA \cos\theta = \frac{q}{\varepsilon_0}$$



As the electric field is radial, so the electric lines of force leave the Gaussian surface normally at all points. Therefore, the electric field intensity **E** and surface area element **dA** are in same direction i.e., $\theta = 0°$.

$$\oint E\, dA = \frac{q}{\varepsilon_0}$$

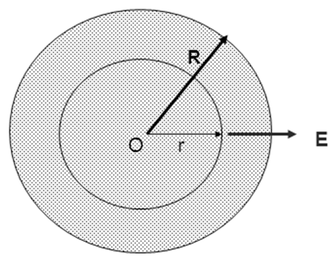
$$E \oint dA = \frac{q}{\varepsilon_0} \qquad \therefore E \text{ is constant}$$

$$E(4\pi r^2) = \frac{q}{\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} \qquad \text{---------------------- (12)}$$

Thus for all points outside the spherical charge distribution, the electric field has the same value as if the charge is concentrated at the center of sphere.

Case 2: When the point '*P*' is inside the sphere of charge



Consider a spherical distribution of charge of radius 'R' with the uniform volume charge density 'ρ'. The total charge in this uniform charge distribution is

$$\rho = \frac{q}{V}$$

$$q = \rho V$$

$$q = \rho\left(\frac{4}{3}\pi R^3\right) \qquad \text{---------------------- (13)}$$

We want to find the electric field at point 'P' at a distance $r < R$ from the center of charged sphere. For this we consider a spherical Gaussian surface of which passes through point P as shown in the figure below.

Let the Gaussian surface encloses the charge $q' < q$ given by

$$q' = \rho\left(\frac{4}{3}\pi r^3\right) \qquad \text{---------------------- (14)}$$

Dividing eq. (12) and (13), we get

$$\frac{q'}{q} = \frac{\rho\left(\frac{4}{3}\pi r^3\right)}{\rho\left(\frac{4}{3}\pi R^3\right)}$$

$$\frac{q'}{q} = \frac{r^3}{R^3}$$

$$q' = \frac{r^3}{R^3}\, q \qquad \text{-------------} \quad \textbf{(15)}$$

According to Gauss's law

$$\oint \mathbf{E.\,dA} = \frac{q'}{\varepsilon_0}$$

$$\oint E\, dA = \frac{q'}{\varepsilon_0} \qquad \therefore E \text{ is directed radially outward}$$

$$E\oint dA = \frac{q'}{\varepsilon_0} \qquad \therefore E \text{ is constant}$$

$$E(4\pi r^2) = \frac{q'}{\varepsilon_0}$$

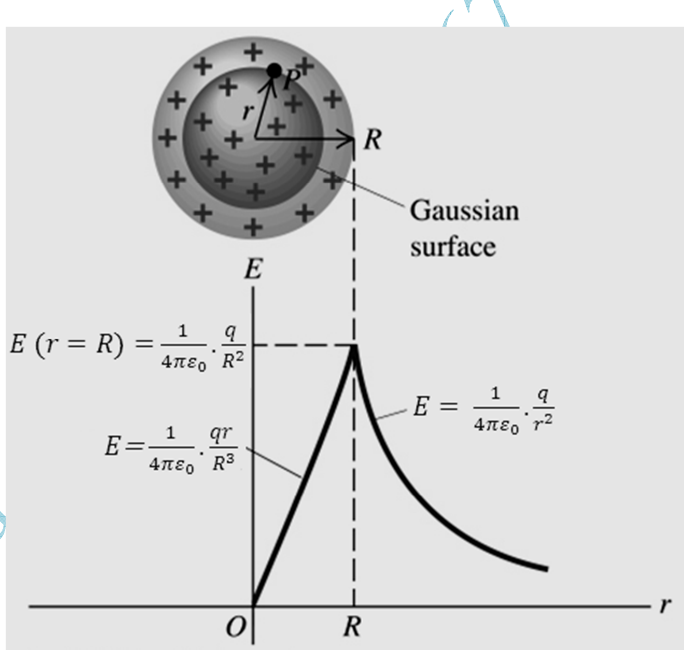
$$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q'}{r^2}$$

Putting the value of q' from eq. (14), we get

$$E = \frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\cdot\frac{r^3}{R^3}\, q$$

$$E\,(r = R) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{R^2} \qquad \text{----} \ \textbf{(16)}$$

The equations (12) and (16) describe the dependence of electric field strength as a function of radial distance 'r' from the center of this charge distribution, the graphical representation of which is given below

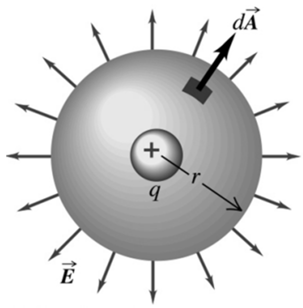


29.6 Deduction of Coulomb's Law from Gauss's Law

Coulomb's law can be deduced from Gauss's law under certain symmetry consideration. Consider positive point charge 'q'. In order to apply the Gauss's law, we assume a spherical Gaussian surface as shown in the figure below.

Considering the integral form of Gauss's law,

$$\oint \mathbf{E.\,dA} = \frac{q}{\varepsilon_0}$$



Because the both vectors **E** and **dA** are directed radially outward, so the quantity **E.dA** becomes simply *E dA*. Therefore,

$$\varepsilon_0 \oint E\, dA = q$$

As E is constant for all the points on the sphere,

$$\varepsilon_0 E \oint dA = q$$

∴ For spherical symmetry $\oint dA = 4\pi r^2$

$$\varepsilon_0 E(4\pi r^2) = q$$

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$ ---------------------- **(16)**

Eq. (6) gives the magnitude of electric field intensity E at any point which is at the distance 'r' from an isolated point charge 'q'.

From the definition of electric field intensity, we know that

$$F = q_0 E$$

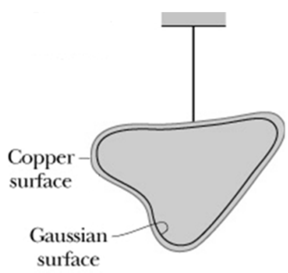
Where q_0 is the point charge placed at a point at which the value of electric field intensity has to be determined. Therefore

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q q_0}{r^2}$$ ---------------------- **(17)**

This is the mathematical form of Coulomb's law.

29.7 Prove that "An excess charge added to the isolated conductor moves entirely to its outer surface. None of the excess charge is found within the body of conductor".

Consider an isolated conductor (lump of copper) is hanging from a silk thread and carrying a net positive charge 'q' as shown in the figure below. The Gaussian surface lies inside the actual surface of the conductor.



Under the equilibrium conditions, the electric field inside the conductor must be zero. If it were not so, the field would exert the force on conduction electrons and the internal currents would be setup.

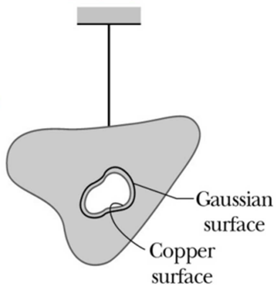
But there is no experimental evidence of such internal currents in isolated conductors. And if the extra charge is added to the surface, it redistribute itself on the surface in such a way that the electric field inside the conductor vanish.

If E is zero everywhere inside the conductor, it must be zero at all the points of Gaussian surface. This means that the flux through Gaussian surface must be zero. Gauss's law $(\Phi_e = \frac{q}{\varepsilon_0})$ then tells that the net charge inside the Gaussian surface must also be zero.

If the added charge is not inside the Gaussian surface, it can only be outside that surface. And the added charge must lie on the actual outer surface of the conductor.

29.8 Prove that the formation of cavity by cutting a natural material from the conductor does not change the distribution charge or pattern of electric field.

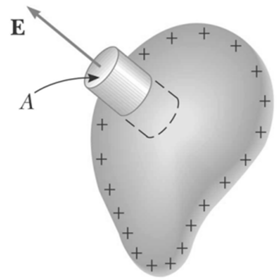
Consider an isolated conductor hanging from a silk thread carrying a net positive charge 'q'. Suppose a cavity is produced inside the conductor as shown in the figure below.



We draw a Gaussian surface around the cavity inside the conductor. Because E is zero inside the conductor, there is no flux though the Gaussian surface. So by Gauss's law, there is no net charge inside the Gaussian surface. So the total charge remains on the outer surface of conductor.

29.9 The External Electric Field

The electric field outside a charged isolated conductor can be find out by Gauss's law by considering the cylindrical Gaussian surface as shown in the figure below.



The flux through the interior end cap is zero, because E = 0 for all interior points of conductor. The flux through the cylindrical walls is also zero because the lines of E are parallel to the surface. But the flux through the outer cap will not be zero.

The total flux can then be calculated as

$$\Phi_e = \oint \mathbf{E}.d\mathbf{A} = \oint_{\substack{\text{outer}\\\text{cap}}} \mathbf{E}.d\mathbf{A} + \oint_{\substack{\text{inner}\\\text{cap}}} \mathbf{E}.d\mathbf{A} + \oint_{\substack{\text{Cylendrical}\\\text{walls}}} \mathbf{E}.d\mathbf{A}$$

$$\Phi_e = EA + 0 + 0 = EA \qquad \text{---------------------} \quad \textbf{(18)}$$

According to Gauss's law

$$\Phi_e = \frac{q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \qquad \text{---------------------} \quad \textbf{(19)}$$

$$\therefore q = \sigma A$$

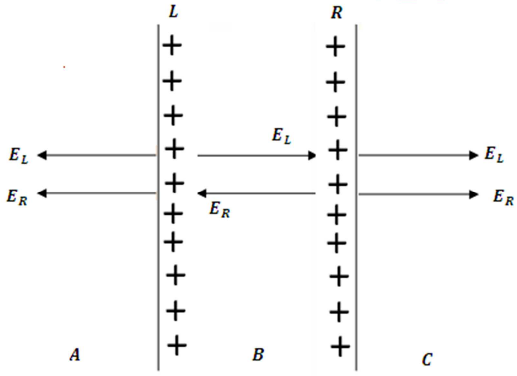
Comparing Eq. (8) and (9), we get

$$E = \frac{\sigma}{\varepsilon_0}$$

This showed that the electric field intensity at any point is doubled than the value of E for an infinite sheet of charge.

A sheet of charge can be constructed by spraying charges on one side of thin layer of plastic. But if we want to charge a thin conducting layer of same dimensions, then we must have to provide twice amount of charge as compared to the case of an insulator sheet.

We can understand the electric field of a thin conducting sheet by as shown in the figure below.



If we consider each face of the conductor as a sheet of charge having electric field $E = \frac{\sigma}{2\varepsilon_0}$, then electric fields in regions A and C will reinforce each other. But in region B, the fields due to left and right fields will cancel out each other.

The electric field inside the region A and C is

$$E = \boldsymbol{E_L} + \boldsymbol{E_R} = \frac{\sigma}{2\varepsilon_0} + \frac{\sigma}{2\varepsilon_0}$$

$$E = \frac{\sigma}{\varepsilon_0}$$

The electric field inside the region B is

$$E = \boldsymbol{E_L} + \boldsymbol{E_R} = \frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0}$$

$$E = 0$$

Hence, the interior of the conductor is charge free region.