ELECTRIC POTENTIAL

30.1 Potential Difference

Potential difference 'ΔV' between two point is defined as "the amount of work done 'ΔW' per unit charge 'q_0' in moving it from one point to the other against the electric field and by keeping the system in equilibrium". Mathematically

$$\Delta V = \frac{\Delta W}{q_0}$$

Suppose a unit positive test charge 'q_0' is moved from one point 'a' to the point 'b' in the electric field '**E**' of a large positive charge 'q' as shown in figure below,
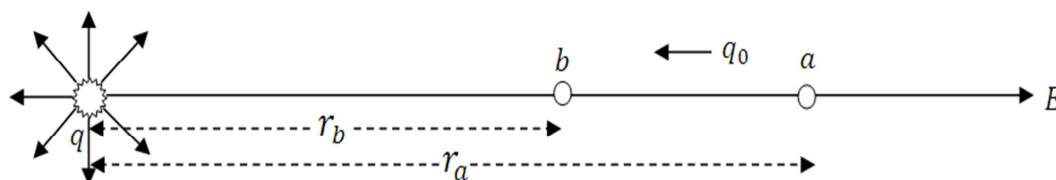
The work done in moving 'q_0' from point 'a' to the point 'b' against the electric field '**E**' is

$$W_{a \to b} = \int_a^b \mathbf{F}.\,\mathbf{dr}$$

The electrical force of magnitude '$\mathbf{F} = -q_0\mathbf{E}$' must have to supplied in order to move 'q_0' against the electric field. Therefore

$$W_{a \to b} = \int_a^b (-q_0\mathbf{E}).\,\mathbf{dr}$$

$$W_{a \to b} = \int_a^b (-q_0\mathbf{E}).\,\mathbf{dr}$$

$$W_{a \to b} = -q_0 \int_a^b \mathbf{E}.\,\mathbf{dr}$$

$$\frac{W_{a \to b}}{q_0} = -\int_a^b \mathbf{E}.\,\mathbf{dr}$$

$$\because \frac{W_{a \to b}}{q_0} = \Delta V = V_b - V_a$$

Therefore, the electrical potential difference between two points in an electrical field will be

$$V_b - V_a = -\int_a^b \mathbf{E}.\,\mathbf{dr} \qquad \text{--------------------} \qquad (1)$$

30.2 Absolute Electrical Potential at a Point

Absolute electric potential at a point is defined as "the amount of work done per unit charge in moving it ffrom infinity to a specific field point against the electric field and by keeping the system in equilibrium".

To find the absolute potential, the reference point is selected at which potential is zero. This point is situated at infinity i.e., out of the electric field. Thus, in equation (1)

$$V_a = V(\infty) = 0$$

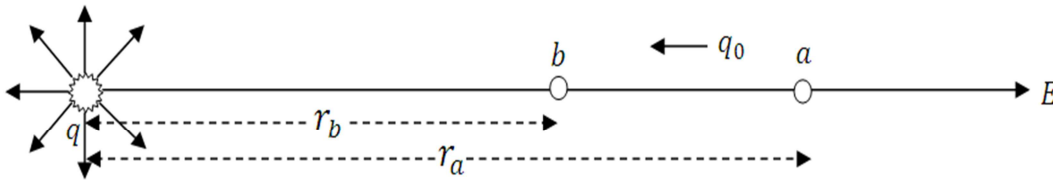$$\Rightarrow V_b - 0 = -\int_{\infty}^{b} \mathbf{E}.\,\mathbf{dr}$$

If the distance from the point 'b' to the charge 'q' is 'r', then in general

$$V(r) = -\int_{\infty}^{r} \mathbf{E}.\,\mathbf{dr}$$

30.3 Expression for the Electric Potential Difference due to a Point Charge

The potential difference between two points is the amount of work done per unit charge 'q_0' in moving it from one point to the other against the electric field 'E'. Mathematically, it is described as

$$V_b - V_a = -\int_{r_a}^{r_b} \mathbf{E}.\,\mathbf{dr}$$

But

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

So,

$$V_b - V_a = -\int_{r_a}^{r_b}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}.\,\mathbf{dr}$$

$$\Rightarrow V_b - V_a = -\frac{q}{4\pi\varepsilon_0}\int_{r_a}^{r_b}\frac{\hat{r}.\,\mathbf{dr}}{r^2}$$

∴E is directed radially outward, therefore $\hat{r}$ // $\mathbf{dr}$

$$\Rightarrow V_b - V_a = -\frac{q}{4\pi\varepsilon_0}\int_{r_a}^{r_b}\frac{\mathbf{dr}}{r^2}$$

$$\Rightarrow V_b - V_a = -\frac{q}{4\pi\varepsilon_0}\left|-\frac{1}{r}\right|_{r_a}^{r_b} = \frac{q}{4\pi\varepsilon_0}\left|\frac{1}{r}\right|_{r_a}^{r_b}$$

$$\Rightarrow V_b - V_a = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_b} - \frac{1}{r_a}\right] \qquad \text{-------------------} \quad (2)$$

This is the expression for the potential difference between two points 'a' and 'b'.

30.4 Expression for the Absolute Electric Potential due to a Point Charge

The electric potential at any point is the amount of work done per unit charge in moving a unit positive charge (test charge) from infinity to that point, against the electric field.

If the point 'a' is at infinity then

$$V_a = V(\infty) = 0 \; ; \; r_a = \infty$$

Putting this value in equation (2), we get

$$\Rightarrow V_b - 0 = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_b} - \frac{1}{\infty}\right] \qquad \therefore \frac{1}{\infty} = 0$$
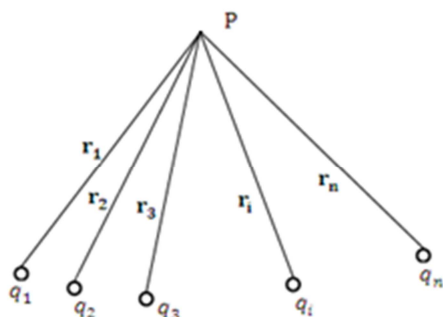
$$\Rightarrow V_b = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_b}\right]$$

In general, the electric potential at point due to a point charge 'q' is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

30.5 Electric Potential Due to a Collection of Point Charges

Let "$q_1, q_2, q_3, \ldots\ldots, q_n$" are the 'n' point charges which are at distances "$r_1, r_2, r_3, \ldots\ldots, r_n$" from a point '$P$', as shown in the figure below

Now if V_1, V_2, V_3, $\ldots\ldots, V_n$ be the electric potential at a field point 'P' due to the point charges q_1, q_2, $q_3, \ldots\ldots, q_n$ respectively. Then, the total electric potential at a field point 'P' due to assembly of 'n' point charges will be;

$$V = V_1 + V_2 + V_3 + \ldots\ldots + V_n \qquad \text{--------------------} \qquad (3)$$

where

$$V_1 = \text{Electric Potential at a Field Point '}P\text{' due to Point Charge '}q_1\text{'} = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1}$$

$$V_2 = \text{Electric Potential at a Field Point '}P\text{' due to Point Charge '}q_2\text{'} = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2}$$

$$V_3 = \text{Electric Potential at a Field Point '}P\text{' due to Point Charge '}q_3\text{'} = \frac{1}{4\pi\varepsilon_0}\frac{q_3}{r_3}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$V_n = \text{Electric Potential at a Field Point '}P\text{' due to Point Charge '}q_n\text{'} = \frac{1}{4\pi\varepsilon_0}\frac{q_n}{r_n}$$

Putting values in equation (3), we get

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_1} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r_2} + \frac{1}{4\pi\varepsilon_0}\frac{q_3}{r_3} + \ldots\ldots + \frac{1}{4\pi\varepsilon_0}\frac{q_n}{r_n}$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \frac{q_3}{r_3} + \ldots\ldots + \frac{q_n}{r_n}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i} \qquad \text{--------------------} \qquad (4)$$

This is the net potential at any point 'P' due to collection of 'n' point charges.
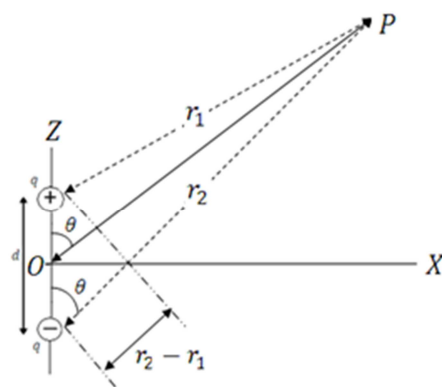
30.6 Electric Potential due to a Dipole

Consider two point charges $+q$ and $-q$ of equal magnitude lying distance 'd' apart as shown in the figure below.

The electric potential at point 'P' is the sum of potential due to '$+q$' and '$-q$'. If V_+ and V_- are the electric potential due to the charges $+q$ and $-q$.

Therefore,

$$V = V_+ + V_-$$
$$V = \frac{1}{4\pi\varepsilon_0}\frac{+q}{r_1} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{r_2}$$
$$V = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$
$$V = \frac{q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$

Normally for a dipole,
$r \gg d$
Therefore,
$r_1 r_2 \approx r^2$
And from figure, $r_2 - r_1 = d\cos\theta$

$$V = \frac{q}{4\pi\varepsilon_0}\left[\frac{d\cos\theta}{r^2}\right]$$
$$V = \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2}$$

$\therefore p = qd$ = Dipole Moment

$$V = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$
$$V = \frac{1}{4\pi\varepsilon_0}\frac{p\,r\cos\theta}{r^3}$$

$\therefore Multiplying\ and\ dividing\ by\ r$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}.\mathbf{r}}{r^3}$$

This is the expression of electric potential at any point 'P' due to a dipole.

30.7 Electric Potential of Continuous Charge Distribution

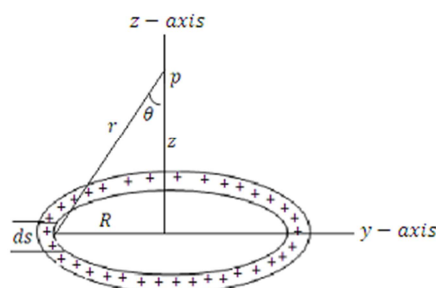### 30.7.1 Electric Potential due to a Ring of Charge

Consider a ring of positive charge of radius 'R' as shown in the figure. We want to find out the electric potential at point 'P' which is at the distance 'z' from the plane of ring.

As the charge is distributed uniformly over it, so it has constant linear charge density λ. For an infinitesimal length element 'ds' of ring,

$$\lambda = \frac{dq}{ds}$$
$$dq = \lambda\,ds$$

The electric potential due to the charge dq at point 'P' is given by;

$$dV = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda \, ds}{r}$$

$$dV = \frac{1}{4\pi\varepsilon_0} \frac{\lambda \, ds}{\sqrt{z^2 + R^2}}$$

$$\therefore From \; Figure, r = \sqrt{z^2 + R^2}$$

The total value of electric potential is

$$V = \int dV = \frac{1}{4\pi\varepsilon_0} \int \frac{\lambda \, ds}{\sqrt{z^2 + R^2}}$$

$$V = \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{z^2 + R^2}} \int \lambda \, ds$$

$$\therefore \int \lambda \, ds = q$$

Therefore,

$$V = \frac{1}{4\pi\varepsilon_0} \frac{q}{\sqrt{z^2 + R^2}}$$

This is the expression of electric potential at any point 'P' due to a ring of charge.

30.7.2 Electric Potential due to a Disk of Charge

Consider a circular disk of uniform surface charge density as shown in the figure below. We want to find out the electric potential at point 'P' which is at the distance 'z' from the plane of disk.

Consider a small element of the disk in the ring shape of radius 'ω' and width 'dω'. If 'dq' is the charge on this element of ring, then

$$\sigma = \frac{dq}{dA}$$

$$\therefore where \; dA \; is \; the \; is \; the \; area \; of \; length \; element$$

$$dq = \sigma \, dA$$

$$dq = \sigma \, (2\pi\omega \, d\omega)$$

$$\therefore dA = 2\pi\omega \, d\omega$$

We know the electric potential due to the ring of charge which has the radius ω is given by;

$$dV = \frac{1}{4\pi\varepsilon_0} \frac{dq}{r}$$

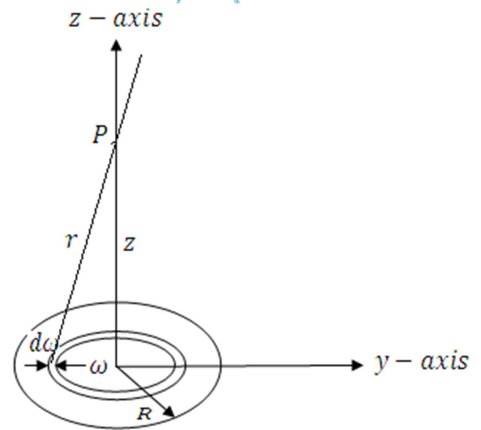$$dV = \frac{1}{4\pi\varepsilon_0} \frac{\sigma \, (2\pi\omega \, d\omega)}{z^2 + \omega^2}$$

The electrical potential at point 'P' due to whole disk is

$$V = \frac{1}{4\pi\varepsilon_0} \int_{\omega=0}^{\omega=R} \frac{\sigma \, (2\pi\omega \, d\omega)}{z^2 + \omega^2}$$

$$V = \frac{\sigma}{4\varepsilon_0} \int_{\omega=0}^{\omega=R} (z^2 + \omega^2)^{-\frac{1}{2}} \, (2\omega) \, d\omega$$

$$V = \frac{\sigma}{4\varepsilon_0} \left| \frac{(z^2 + \omega^2)^{-\frac{1}{2}+1}}{\frac{1}{2}} \right|_0^R$$

$$V = \frac{\sigma}{2\varepsilon_0} \left[(z^2 + R^2)^{\frac{1}{2}} - (z^2)^{\frac{1}{2}} \right]$$

$$V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{z^2 + R^2} - z\right]$$

This is the expression of electric potential at point 'P' due to disk of charge.

Special Case:

If $z \gg R$, then

$$\sqrt{z^2 + R^2} = (z^2 + R^2)^{\frac{1}{2}}$$

$$= z\left[1 + \frac{R^2}{z^2}\right]^{\frac{1}{2}}$$

$$= z\left[1 + \frac{1}{2}\frac{R^2}{z^2} + \,...\,...\,...\right]$$

$$\Rightarrow \sqrt{z^2 + R^2} = \left[z + \frac{R^2}{2z}\right] \qquad \therefore \text{ Neglecting Higher Terms}$$

Therefore,

$$V = \frac{\sigma}{2\varepsilon_0}\left[z + \frac{R^2}{2z} - z\right] = \frac{\sigma R^2}{4\varepsilon_0 \, z}$$

$$V = \frac{\sigma \, \pi R^2}{4\pi\varepsilon_0 \, z} = \frac{\sigma\,(\pi R^2)}{4\pi\varepsilon_0 \, z}$$

$$\therefore \sigma\,(\pi R^2) = q \; (total \; charge \; on \; disk)$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{z}$$

Thus, the disk of charge behave like a point charge for the case $z \gg R$.

30.8 Equipotential Surfaces

      If all the points on a surface have the same value of electric potential, then, it is known as equipotential surface. The examples of some equipotential surfaces are given below

- In case of uniform electric field, the equipotential surfaces are the planes as shown by the dashed line in fig. (a). The potential at points B_1, B_2 and B_3 is same. So, no work will be done in moving a test charge from B_1 to B_2 or B_3.

- The concentric spheres about a point charge '$+q$' are the equipotential surfaces as shown in the fig. (b) by dashed lines.

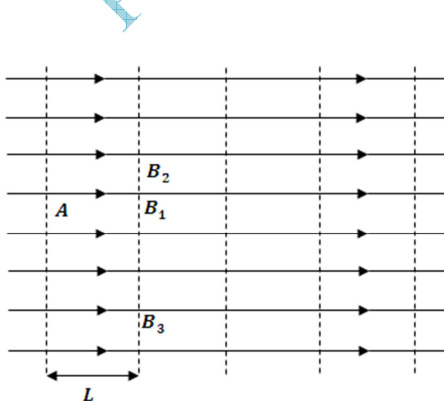- The equipotential surfaces of a dipole are shown by the dashed lines in fig. (c)
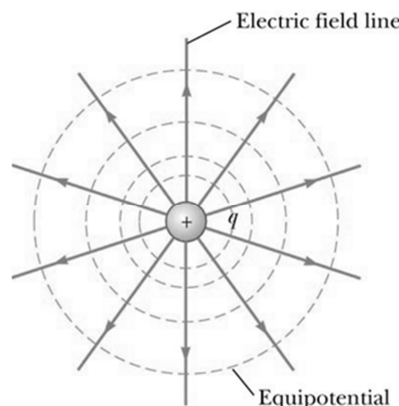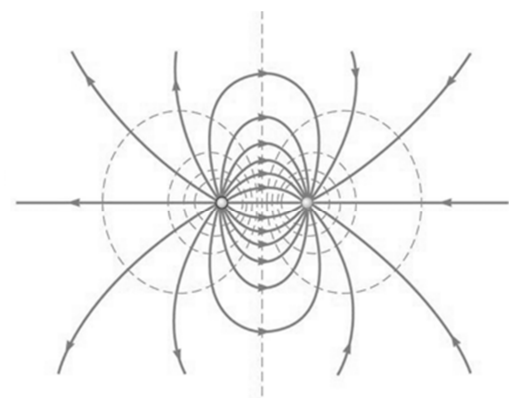

Fig. (a)   Fig. (b)   Fig. (c)

30.9 Calculating Electric Field from the Electric Potential

Consider a test charge 'q_0' is displaced in an electric field form point 'P' to point 'Q'. If the potential difference between the points 'P' and 'Q' is dV, then the work done 'dW' by the electric field is

$$dW = -q_0 dV \quad \text{------------------- (1)}$$

If the test charge is placed in an electric field, then the electrical force on the test charge is

$$\mathbf{F} = q_0 \mathbf{E}$$

So, the corresponding work done is

$$dW = \mathbf{F}.\mathbf{ds}$$
$$= q_0 \mathbf{E}.\mathbf{ds} \quad \text{------------------- (2)}$$

Comparing eq. (1) and (2)

$$q_0 \mathbf{E}.\mathbf{ds} = -q_0 dV$$
$$q_0 E\, ds\, cos\theta = -q_0 dV$$
$$E\, cos\theta = -\frac{dV}{ds}$$

Consider '$E\, cos\theta = E_s$' which is the component of '$\mathbf{E}$' along '$\mathbf{ds}$'

$$E_s = -\frac{dV}{ds}$$

There is only one direction in which the rate of change of electric potential w r t position is maximum, which is direction of electric field. Thus

$$E_s = -\left(\frac{dV}{ds}\right)_{max}$$

The maximum value of $\frac{dV}{ds}$ at a given point is called the 'potential gradient'. For any arbitrary direction of '$\mathbf{ds}$'

$$E_x = -\frac{\partial V}{\partial x}\, ; E_y = -\frac{\partial V}{\partial y}\, ; E_z = -\frac{\partial V}{\partial z}$$

Therefore,

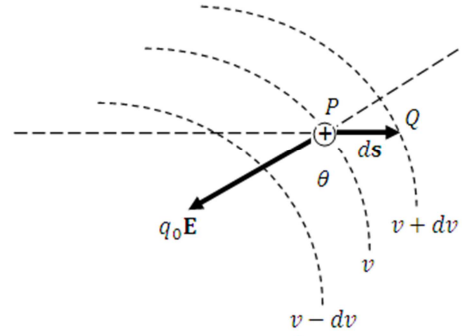$$\mathbf{E} = E_x \hat{\imath} + E_y \hat{\jmath} + E_z \hat{k}$$
$$\mathbf{E} = -\left(\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}\right)$$
$$\mathbf{E} = -\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)V$$
$$\mathbf{E} = -\nabla V$$
$$\mathbf{E} = -\text{grad}\, V$$

This expression states that electric field is negative gradient of electric potential.

Question: Find the expression of electric field from electric potential due to a point charge.

Solution: the electric potential due to a point charge is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{x^2+y^2+z^2}} \qquad\qquad \therefore \sqrt{x^2+y^2+z^2}$$

$$\text{grad } V = (\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k})\,V$$

$$= (\frac{\partial V}{\partial x}\hat{\imath} + \frac{\partial V}{\partial y}\hat{\jmath} + \frac{\partial V}{\partial z}\hat{k}) \text{ -------------------- } (3)$$

Now consider

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left[\frac{1}{4\pi\varepsilon_0}\frac{q}{\sqrt{x^2+y^2+z^2}}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{-\frac{1}{2}(2x)}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial V}{\partial x} = -\frac{q}{4\pi\varepsilon_0}\frac{x}{(x^2+y^2+z^2)^{3/2}} = -\frac{q}{4\pi\varepsilon_0}\frac{x}{r^3}$$

Similarly

$$\frac{\partial V}{\partial y} = -\frac{q}{4\pi\varepsilon_0}\frac{y}{(x^2+y^2+z^2)^{3/2}} = -\frac{q}{4\pi\varepsilon_0}\frac{y}{r^3}$$

$$\frac{\partial V}{\partial z} = -\frac{q}{4\pi\varepsilon_0}\frac{z}{(x^2+y^2+z^2)^{3/2}} = -\frac{q}{4\pi\varepsilon_0}\frac{z}{r^3}$$

Putting these values in eq. (3)

$$\text{grad } V = -\frac{q}{4\pi\varepsilon_0}\left[\frac{x\hat{\imath}+y\hat{\jmath}+z\hat{k}}{r^3}\right]$$

$$\therefore \mathbf{r} = x\hat{\imath}+y\hat{\jmath}+z\hat{k}$$

$$\text{grad } V = -\frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{r}}{r^3}\right]$$

$$\text{grad } V = -\frac{q}{4\pi\varepsilon_0}\left[\frac{r\,\hat{r}}{r^3}\right]$$

$$\therefore \mathbf{r} = r\,\hat{r}$$

$$\text{grad } V = -\frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{r}}{r^2}\right]$$

$$-\text{grad } V = \frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{r}}{r^2}\right] \qquad \therefore \mathbf{E} = -\text{grad } V$$

Therefore

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{r}}{r^2}\right]$$

which is the expression of electric field intensity due to a point charge.

30.10 Electric Potential and Electric Field Inside and Outside an Isolated Conductor

If the excess of charge is placed on an isolated conductor, then it moves entirely on to the outer surface of the conductor. In equilibrium, none of the charge is inside the body of conductor or on any interior surface (even the conductor has internal cavities). This property of the conductors can be stated in terms of electrical potential as

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"An excess of charge placed on an isolated conductor distributes itself on the surface so that all points of conductor come to the same potential".

(a)

Now consider an isolated conductor in the form of a spherical shell having uniform surface charge density. In this case, the electric field is radial and its value is maximum on the surface of the spherical shell. Also, it decreased as we move away from the sphere because

$$E \propto \frac{1}{r^2}$$

(b)

The electric field at any point inside sphere is zero.

The electric potential decreases as we move away from the sphere, because

$$V \propto \frac{1}{r}$$

(c)

The value of electric potential is maximum at the surface of sphere and it remains constant at every point inside the sphere. This is so because if a test charge is pushed inside the sphere through the hole, then it experiences no force and no work will be done. Hence the electric potential at all the points inside the sphere is same.

As the electric potential inside and on the surface of sphere has the same constant value, its derivative will be zero.

As

$$V = Constant$$

$$\therefore E = -\frac{dV}{ds} = -\frac{d}{ds}(constant) = 0$$

So the electric field inside the sphere is zero and it has maximum value at the surface of sphere. The variation of electric potential with respect to 'r' and variation electric field with respect to 'r^2' is shown in the figure below.