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Chapter 2

WAVES

1. Mechanical Waves:

"Those waves which require a material medium for their propagation are called Mechanical waves."

They carry energy and momentum with them e.g sound waves and water waves. Mechanical waves are of two types.

(i) Longitudinal waves (ii) Transverse waves.

Such waves in which particles of the medium vibrate ⊥ to the the direction of propagation of waves are called Transverse waves. e.g If a string is tied to a hook from one end and the free end is displaced up and down then transverse waves are produced. The disturbance moves along the string but the particles vibrate ⊥ to the direction of propagation of disturbance. Water waves are also transverse in nature.

All the electromagnetic waves are transverse waves.

"Such waves in which particles of the medium vibrate along the direction of propagation of waves are called Longitudinal waves." e.g when a spring under tension is made to vibrate back and forth at one end, a longitudinal wave is produced. The coils of the spring vibrate back and forth ∥ to the direction of propagation of disturbance.

Sound waves in a gas are longitudinal waves or compressional waves. Waves can also be classified as one, two and three dimensional according to the number of dimensions in which they propagate energy. e.g

(i) Waves moving along the string or spring are one dim...
(ii) Water waves produced by throwing a pebble into stationary pond are two dimensional.
(iii) Sound waves and light waves travelling radially outward from the source are three dimensional.

When a stone is dropped in a still water, a circular wave spreads from the point where the stone strikes the water. Along a given circular wave, all the points are in the same state of vibration. "The path (locus) of all points having the same state of vibration is called a wave front."

"A line drawn ⊥ to the wave front is called a ray or the direction in which a wave travels is called a ray."

The wave front may be spherical or plane. e.g Light travels from a source in the form of a expanding sphere is called spherical wave front. At a very large distance from the source a part of the spherical wave front becomes straight called plane wave front.

2. Travelling Waves:

A system which has no definite outer boundries is called an open system. In this system the energy fed from some outside source does not remain with in the system but flow out of it in the form of waves called Travelling waves.

"The waves produced by a source coupled to an open system are called Travelling waves."

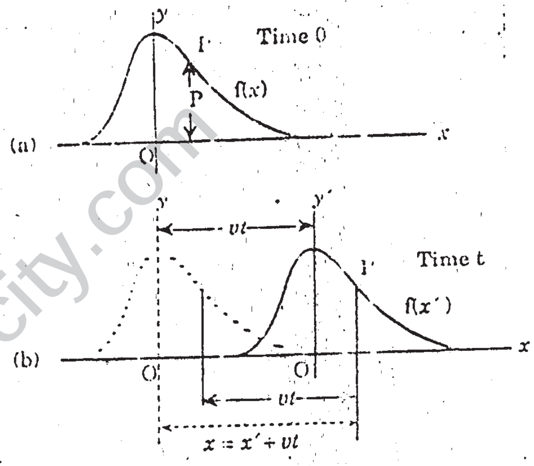
These waves travel away from the source which produces them. They carry energy and momentum with them.

Consider a string whose one end is fixed to a hook and other end is held in hand. If we move the free end up and down at regular intervals, a series of identical waves travels along the string. This series of identical waves forms a travelling wave.

Consider a transverse pulse on a long stretched string at $t=0$. We suppose the shape of pulse remains unchanged as it travels the pulse lies in xy-plane. Let the pulse be travelling in +ve x-direction with speed v.

After some time 't' the pulse coverse a distance vt. as shown in fig. (b) and its shape is the same as at $t=0$.

The y-coordinate represents the transverse displacement of a point on the string which depends both in on position x and time t. i.e 'y' is a function of x and t.



$$\therefore\ y = y(x, t)$$

So the wave form of the pulse at $t=0$ is given by

$$y(x, 0) = f(x) \quad -(1)$$

where the function 'f' gives the shape of the wave.

After time 't' the shape of the wave remains unchaged. So after time 't' the wave form is represented by the same function 'f'.

Now consider two frames of reference S and S'. Frame S is at rest and S' is moving along with the wave with speed V.

$\therefore$ According to moving frame S', the wave form of the pulse is given by the function $f(x')$.

Now x and x' are related as,

$$x' = x - vt \qquad \because x = x' + vt$$

where x' is the displacement of the pt 'P' along x-axis w.r.t. S' and x is the displacement of point P w.r.t. S.

So the wave form of the pulse at time t is given by

$$y(x,t) = f(x').$$

$$y(x,t) = f(x - vt) \quad \text{--- (3)}$$

From equations (1) and (3) we see that we can change function of any shape into a wave travelling in the +ve x-direction simply by replacing x by $(x - vt)$ in the function $f(x)$.

If a wave has to keep its shape unchanged then the y-coordinate of point P increases as t increases such that $(x - vt)$ remains constant.

$$\text{i.e.} \quad x - vt = \text{constant} \quad \text{--- (4)}.$$

Phase Velocity (V)

"The velocity of the travelling wave in any medium is called phase velocity. It is obtained by differentiating eq. (4) w.r.t. 't'.

$$\text{i.e.} \quad \frac{d}{dt}(x - vt) = \frac{d}{dt}(\text{constt})$$

$$\frac{dx}{dt} - v = 0.$$

$$\frac{dx}{dt} = v \quad \text{--- (5)}$$

Eq. (5) shows phase velocity of the wave. It depends on the nature of medium.

If the wave is travelling in -ve x-direction then eq. (3) becomes.

$$y(x,t) = f(x + vt) \quad \text{--- (6)},$$

$\therefore$ Eq. (4) becomes

$$x + vt = \text{constt}.$$

$$\frac{d}{dt}(x + vt) = \frac{d}{dt}(\text{constt})$$

$$\frac{dx}{dt} + v = 0.$$

$$\frac{dx}{dt} = -v.$$

where -ve sign shows that wave is moving towards -ve

... holds for transverse as well as for

where x' is the displacement of the pt 'P' along x-axis w.r.t. S' and x is the displacement of point P w.r.t. S.
So the wave form of the pulse at time t is given by

$$y(x,t) = f(x').$$

$$y(x,t) = f(x - vt) \quad \text{— ③}$$

From equations ① and ③ we see that we can change function of any shape into a wave travelling in the +ve x-direction simply by replacing x by $(x - vt)$ in the function $f(x)$.
If a wave has to keep its shape unchanged then the y-coordinate of point P increases as t increases such that $(x - vt)$ remains constant.

$$\text{i.e.} \quad x - vt = \text{constant} \quad \text{— ④}.$$

Phase Velocity (V)

"The velocity of the travelling wave in any medium is called phase velocity. It is obtained by differentiating eq. ④ w.r.t. 't'.

$$\text{i.e.} \quad \frac{d}{dt}(x - vt) = \frac{d}{dt}(\text{constt})$$

$$\frac{dx}{dt} - v = 0.$$

$$\frac{dx}{dt} = v \quad \text{— ⑤}$$

Eq. ⑤ shows phase velocity of the wave. It depends on the nature of medium.
If the wave is travelling in -ve x-direction then eq. ③ becomes.

$$y(x,t) = f(x + vt) \quad \text{— ⑥},$$

$\therefore$ Eq. ④ becomes

$$x + vt = \text{constt}.$$

$$\frac{d}{dt}(x + vt) = \frac{d}{dt}(\text{constt})$$

$$\frac{dx}{dt} + v = 0.$$

$$\frac{dx}{dt} = -v.$$

where -ve sign shows that wave is moving towards -ve
[illegible] holds for transverse as well as for

for longitudinal waves.

3. Sinusoidal Waves:

Consider a transverse wave of sinusoidal shape. Suppose at time t=0 the wave train is moving along x-axis and is given by $y(x,0) = y_m \sin kx$.

where $k = \frac{2\pi}{\lambda}$ is the wave number having unit rad/m.

$$\therefore \quad y(x,0) = y_m \sin\left(\frac{2\pi}{\lambda}\right)x \quad \text{—} (1)$$

Here y_m is the maximum displacement and is called the amplitude of the wave and λ is ~~called~~ the wave length.

Thus a wave travelling in the +ve x-direction with phase velocity V at any time 't' is given by.

$$y(x,t) = y_m \sin\left(\frac{2\pi}{\lambda}\right)(x - Vt) \quad \text{—} (2)$$

Now time period of wave T is defined as the time during which the wave covers a distance 'λ' with phase velocity V.

$$\therefore \quad S = VT.$$

$$\lambda = VT$$

$$\text{or} \quad V = \frac{\lambda}{T}$$

$\therefore$ Eq. (2) becomes

$$y(x,t) = y_m \sin\left(\frac{2\pi}{\lambda}\right)\left(x - \frac{\lambda t}{T}\right)$$

$$y(x,t) = y_m \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) \quad \text{—} (3)$$

This shows that the transverse displacement 'y' at any time 't' has the same value at x, $x+\lambda$, $x+2\lambda$ and so on. e.g

$$y(x,t) = y_m \sin 2\pi\left(\frac{x+\lambda}{\lambda} - \frac{t}{T}\right)$$

$$= y_m \sin 2\pi\left(\frac{x}{\lambda} + 1 - \frac{t}{T}\right)$$

$$= y_m \sin\left(\frac{2\pi x}{\lambda} + \frac{2\pi}{T}t + 2\pi\right)$$

$$y(x,t) = y_m \sin\left(\frac{2\pi x}{\lambda} - \frac{2\pi}{T}t\right) \qquad \because \sin\left(\frac{2\pi x}{\lambda} + 2\pi\right) = \sin\frac{2\pi x}{\lambda}$$

$$\because \frac{2\pi}{\lambda} = k \quad \text{and} \quad 2\pi/T = \omega$$

$$\therefore \quad y(x,t) = y_m \sin(kx - \omega t) \quad \text{—} (A)$$

Similarly 'y' at any given position has the same value at time t, t+T, t+2T and so on. e.g putting t = t+T in eq. ③ we get

$$y(x,t) = y_m \sin 2\pi\left(\frac{x}{\lambda} - \frac{t+T}{T}\right)$$

$$= y_m \sin 2\pi\left(\frac{x}{\lambda} - \frac{t}{T} - 1\right)$$

$$= y_m \sin\left(\frac{2\pi x}{\lambda} - \frac{2\pi t}{T} - 2\pi\right)$$

$$= y_m \sin\left(\frac{2\pi x}{\lambda} - 2\pi - \frac{2\pi}{T} t\right)$$

$$= y_m \sin\left(\frac{2\pi x}{\lambda} - \omega t\right) \qquad \because \omega = \frac{2\pi}{T}$$

$$y(x,t) = y_m \sin(kx - \omega t) \text{ — (B)} \qquad \because \frac{2\pi}{\lambda} = k.$$

Eqs. (A) and (B) represent the eq. of a travelling wave travelling in +ve x-direction

The eq. of travelling wave in -ve x-direction is given by

$$y(x,t) = y_m \sin(kx + \omega t)$$

Relation b/w Wave number & Angular Freq.

Wave number is given by

$$k = \frac{2\pi}{\lambda} \text{ —(i)}$$

Angular frequency is given by

$$\omega = \frac{2\pi}{T}. \qquad \text{But } \frac{1}{T} = \nu$$

$$\omega = 2\pi\nu \text{ —(ii)}$$

Because $V = \nu\lambda$; $\nu = \frac{V}{\lambda}$.

$\therefore$ Relation (ii) becomes

$$\omega = 2\pi\frac{V}{\lambda}.$$

$$\omega = \frac{2\pi}{\lambda} V.$$

But from (i) $\frac{2\pi}{\lambda} = k$.

$$\therefore \omega = kV.$$

$$\boxed{V = \frac{\omega}{k}}$$

It should be noted that ω and k both are angular quantities.

The unit of k = rad/m.

The unit of ω = rad/sec.

Group Speed and Dispersion:

The speed of wave in any medium is called phase speed. But the term phase speed is used for those waves which keep their shape unchanged while travelling like Sine wave. But in case of other types of waves e.g square wave, Sawtooth wave, we use the term Group speed instead of phase speed.

"Group speed is the speed of group of waves in a complex wave." This is the speed at which energy travels in a real wave.

Now the medium in which velocity of wave depends on wavelength is called Dispersion medium. The waves whose velocity changes with change in wave length are called dispersive waves.

So in a dispersive medium, the phase speed of component waves depends on the wavelength or frequency of the component wave. So in a dispersive medium different component waves have different phase speeds.

But in a non-dispersive medium both the phase speed of light waves (complex wave) is equal to phase speed of component waves in the complex wave. So in a non dispersive medium all component waves travel with the same speed e.g sound waves in air and light waves in air.

However in a dispersive medium like glass the group speed and phase speed of light waves become different.

Wave Speed -

The wave speed means the phase speed of a Sin wave or group speed of a wave in non dispersive medium. In non dispersive medium the speed is independent of frequency or wave length.

The wave speed of a wave in a medium depends on the propolion of a medium. Here we will find the speed of a transverse wave in a stretched string.

The wave speed can be calculated by two ways.

(i) By Dimensional Analysis. (ii) By Mechanical Analysis.

But we shall study only mechanical analysis.

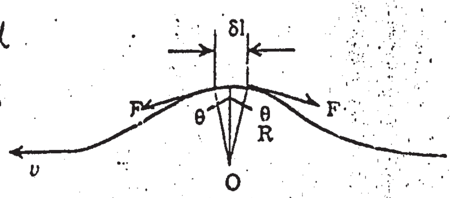
Mechanical Analysis:-

Consider a stretched string in which a transverse wave is produced which is moving with speed V. Let us take a small element of the string of length 'δl'. This element forms an arc of circle. Let R be the radius of the circle.

If μ is the linear mass density (mass per unit length) of the string then mass of element of string is

$$\delta m = \mu \delta l.$$

Let F be the tension at the two ends of the element. This tension is along the tangent at each end of the string element. The horizontal components of F cancel each other and the vertical components are added up.



$\therefore$ Total vertical force is $F_{\perp} = F\sin\theta + F\sin\theta$

$$F_{\perp} = 2F\sin\theta.$$

For small θ ; $\sin\theta \approx \theta$.

$$\therefore \quad F_{\perp} = 2F\theta.$$

$$F_{\perp} = F(2\theta). \quad - \text{①}$$

By using the relation $S = r\theta$.

$$\delta l = R(2\theta).$$

$$2\theta = \frac{\delta l}{R}. \quad - \text{②}$$

Putting ② in ① we get

$$F_{\perp} = F\left(\frac{\delta l}{R}\right) - \text{③}.$$

But $F_{\perp}$ is directed towards the 2θ of circle of arc. So this becomes the centripetal force.

$$\therefore \quad F_{\perp} = \frac{\delta m v^2}{R} \quad - \text{④}$$

Comparing ③ and ④ we get,

$$\frac{\delta m v^2}{R} = F\frac{\delta l}{R}.$$

$$\delta m v^2 = F\delta l.$$

$$v^2 = F\frac{\delta l}{\delta m}.$$

But $\frac{\delta m}{\delta l} = \mu$

$$\therefore \frac{\delta l}{\delta m} = \frac{1}{\mu}.$$

$$\therefore \quad v^2 = \frac{F}{\mu}.$$

$$v = \sqrt{\frac{F}{\mu}}$$

This is the expression for wave speed for small transverse displacement.

Sample Problem - 1

A transverse sinusoidal wave is generated at one end of a long horizontal string by a bar that moves the end up and down through a distance of 1.30 cm. The motion is continuous and is repeated regularly 125 times per second. (a) If the string has a linear density $\mu = 0.251$ kg/m and is kept under a tension of 96 N, find the amplitude, frequency, speed and wavelength of wave motion. (b) Assuming the wave moves in the +ve x-direction and that at $t = 0$, the element of the string at $x = 0$ is at its equilibrium position $y = 0$ and moving downward, find the equation of the wave.

Sol:- $\mu = 0.251$ kg/m ; $F = 96$N.

Total distance moved by the bar $= 1.30$ cm

$\therefore$ Distance b/w mean and extreme position $= \frac{1}{2}(1.30)$

$= 0.65$ cm.

(a) $\therefore$ Amplitude $= y_m = 0.65$ cm. Ans.

As the motion is repeated 125 times per second

$\therefore$ Frequency $= \nu = 125$ Hz. Ans.

Now the wave speed is given by,

$$V = \sqrt{\frac{F}{\mu}}$$

$$= \sqrt{\frac{96}{0.251}} = 19.556.$$

$$\boxed{V = 19.6 \text{ m/s}} \text{ Ans.}$$

The wave length is given by.

$$V = \nu\lambda.$$

$$\lambda = \frac{V}{\nu} = \frac{19.6}{125}.$$

$$\lambda = 0.156 \text{ m}$$

$$\boxed{\lambda = 15.6 \text{ cm}} \text{ Ans.}$$

(b) The eq. of wave $= ?$

The equation of motion of Sinwave moving in x-direction is given by

$$y(x,t) = y_m \sin(kx - \omega t) \quad \text{— (1)}$$

As $\boxed{y_m = 0.65 \text{ cm}}$ —(a)

Now $K = \frac{2\pi}{\lambda} = \frac{2 \times 3.14}{0.156} = 40.3$ rad/m.

$$\boxed{K = 0.403 \text{ rad/cm}} \text{ — (b).}$$

Now $V = \frac{\omega}{K}$.

$$\omega = VK.$$

$$\omega = 19.6 \times 40.3 = 789 \text{ rad/sec.}$$

$$\boxed{\omega = 789 \text{ rad/sec}} \text{ — (c).}$$

Putting the values of y_m, K and ω from (a), (b) and in ① we get,

$$y(x,t) = 0.65 \sin(0.403x - 789t).$$

This is the equation of motion of wave.

Sample Problem 2

As the wave of problem (2.1) passes along the string (a) Find the expressions for the velocity and acceleration of any particle located at $x = 0.245$ m (b) Evaluate the transverse displacement, velocity and acceleration of this particle at $t = 1.50$ sec.

Sol (a) $x = 0.245m = 24.5$ cm, ∵ From S.Problem 1.

$v = ?$; $a = ?$ $\omega = 789$ rad/s

$K = 0.403$ rad/cm

$y_m = 0.65$ cm.

As $y(x,t) = y_m \sin(Kx - \omega t)$

$$\therefore \quad v = \frac{dy}{dt}.$$

$$v = -\omega y_m \cos(kx - \omega t)$$

$$= -789 \times 0.65 \cos(0.403 \times 24.5 - 789t)$$

$$v = -513 \cos(9.87 - 789t) \quad \text{Ans} \quad -①$$

Similarly,

$$a(x,t) = -\omega^2 y_m \sin(kx - \omega t)$$

$$a(x,t) = -(789)^2 \times 0.65 \sin(0.403 \times 24.5 - 789t)$$

$$= -(789)^2 \times 0.65 \sin(9.8735 - 789t)$$

$$\boxed{a(x,t) = -(789)^2 \times 0.65 \sin(9.87 - 789t)} \text{ Ans}$$

(b) As $t = 1.5$ Sec, $y = ?$, $V = ?$, $a = ?$

For transverse displacement

$$y = y_m \sin(kx - \omega t)$$

$$= 0.65 \sin(0.403 \times 24.5 - 789 \times 1.5)$$

$$= 0.65 \sin(9.87 - 1183.5)$$

$$= 0.65 \sin(-1173.63)$$

$$= 0.65 \times 0.97$$

$$\boxed{y = 0.63 \text{ cm}} \text{ Ans.}$$

... velocity from eq. (1) we have

$$V(x,t) = -513\cos(9.87 - 789 \times 1.5)$$
$$= -513\cos(9.87 - 1183.5)$$
$$= -513\cos(-1173.63)$$
$$= -513 \times 0.243.$$

$$\boxed{V(x,t) = -125\ cm/s}$$ Ans.

For acceleration from eq. ② we have;

$$a(x,t) = -(789)^2 \times 0.65 \sin(9.87 - 789 \times 1.5)$$
$$= -(789)^2 \times 0.65 \sin(9.87 - 1183.5)$$
$$= -789 \times 789 \times 0.65 \sin(-1173.63)$$
$$= -789 \times 789 \times 0.65 \times 0.97.$$
$$= -392499$$
$$= -3.92499 \times 10^5\ cm/s^2.$$

$$\boxed{a(x,t) = -3.93 \times 10^5\ cm/s^2}$$ Ans.

The Wave Equation:

In case of S.H.M of a spring-mass system the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{K}{m}x.$$

The solution of this eq. is written as

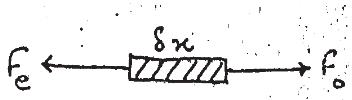
$$x = x_m \cos(\omega t + \varphi)$$

In case of wave motion, the solution can be written as

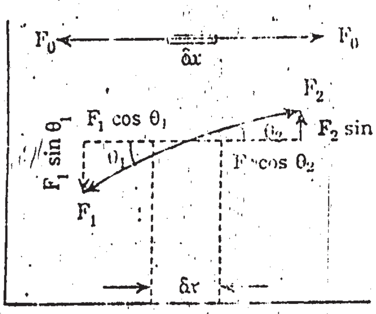
$$f(x) = f(x \pm vt)$$

Consider a string lying along x-axis. let the given string be stretched b/w two fixed supports along x-axis in equilibrium position.

Let F_o be the tension in the string in the equilibrium position.



Consider a small element of the string.



Let δx and δm be the length and mass of this element. Then linear mass density. i.e mass per unit length is

$$\mu = \frac{\delta m}{\delta x}.$$

When the string is plucked (displaced) along y-axis then it does not remain straight but becomes slightly curved as shown. The angles θ_1 and θ_2 are not equal.

Let F_1 and F_2 be the tension at the two ends in the displaced position. Now we resolve the tension into rectangular components. The horizontal components $F_1\cos\theta_1$ and $F_2\cos\theta_2$ being equal and opposite cancel away each other.

The net force acting on the element in the upward direction is given by

$$F_y = F_2 \sin\theta_2 - F_1 \sin\theta_1.$$

$$= F_2 \cos\theta_2 \left(\frac{\sin\theta_2}{\cos\theta_2}\right) - F_1 \cos\theta_1 \left(\frac{\sin\theta_1}{\cos\theta_1}\right)$$

$$= F_2 \cos\theta_2 \tan\theta_2 - F_1 \cos\theta_1 \tan\theta_1.$$

Now $F_2 \cos\theta_2 = F_1 \cos\theta_1 = F$ (say).

$$\therefore F_y = F \tan\theta_2 - F \tan\theta_1.$$

$$= F(\tan\theta_2 - \tan\theta_1).$$

$$F_y = F\,\delta(\tan\theta) \quad -(1).$$

where $\delta(\tan\theta) = \tan\theta_2 - \tan\theta_1$.

But this net force on the element is given by Newton's IInd law of motion as

$$F_y = \delta m\, a_y.$$

$$F_y = \mu\, \delta x\, a_y \quad -(2).$$

$\because \frac{\delta m}{\delta x} = \mu$.

$\delta m = \delta x\, \mu$.

From (1) and (2) we get,

$$F\,\delta(\tan\theta) = \mu\, \delta x\, dy.$$

$$\frac{\delta(\tan\theta)}{\delta x} = \frac{\mu\, dy}{F} \quad \text{— ③.}$$

Now $\tan\theta = \frac{\partial y}{\partial x}$.

and $a_y = \frac{\partial^2 y}{\partial t^2}$.

∴ Eq. ③. becomes.

$$\frac{\delta\left(\frac{\partial y}{\partial x}\right)}{\delta x} = \frac{\mu}{F}\left(\frac{\partial^2 y}{\partial t^2}\right) \quad \text{— ④.}$$

Now $\lim_{\delta x \to 0} \frac{\delta\left(\frac{\delta y}{\delta x}\right)}{\delta x} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial^2 y}{\partial x^2}$ — ⑤.

as the mass element becomes very small.

∴ From equations ④ and ⑤.

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\,\frac{\partial^2 y}{\partial t^2} \quad \text{— ⑥}$$

But $\sqrt{\frac{F}{\mu}} = V$.

$\sqrt{\frac{\mu}{F}} = \frac{1}{V}$.

$\frac{\mu}{F} = \frac{1}{V^2}$.

$$\therefore \boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{V^2}\,\frac{\partial^2 y}{\partial t^2}}.$$

This is the equation of motion of transverse oscillations of the string. Its solution gives the waves on the string. So it is also called Wave equation.

x————x

Solution of Wave Equ.

As the wave equation is given by.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{V^2}\,\frac{\partial^2 y}{\partial x^2} \quad \text{— ①.}$$

We want to find solution of this equation. We know that general formula for travelling wave is

$$y(x,t) = f(x \pm Vt) \quad \text{— ②.}$$

Let us suppose that eq. ② is the solution of eq. ①. Let us prove it that equation ② is the solution of eq. ①.

Proof:- Let $x \pm Vt = z$. —(A)

$\therefore$ Eq. (2) can be written as

$$y(x,t) \; f(z) \quad — (B)$$

Differentiating (B) w.r.t. x

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}\left(f(z)\right)$$

$$= \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} \quad — (a)$$

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial z} \cdot (1)$$

$\therefore$ From (A) $\frac{\partial z}{\partial x} = 1$.

$$\frac{\partial y}{\partial x} = \frac{\partial f}{\partial z}$$

Diff. it w.r.t. x we get

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial z}\right]$$

$$= \frac{\partial}{\partial z}\frac{\partial}{\partial x}[f]$$

$$= \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x}\right]$$

$$= \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial z} \cdot 1\right]$$

$\therefore \frac{\partial z}{\partial x} = 1$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f}{\partial z^2} \quad — (b)$$

Similarly differentiating (B) w.r.t. 't' we get

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}\left[f(z)\right]$$

$$= \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}$$

From eq. (A) $\frac{\partial z}{\partial t} = \pm V$.

$$\therefore \frac{\partial y}{\partial t} = \frac{\partial f}{\partial z}(\pm V)$$

$$\frac{\partial y}{\partial t} = \pm V \frac{\partial f}{\partial z}$$

Diff. w.r.t. 't'

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left[\pm V \frac{\partial f}{\partial z}\right]$$

$$= \frac{\partial}{\partial z}\left[\pm V \frac{\partial}{\partial t} f\right]$$

$$= \frac{\partial}{\partial z}\left[\pm V \frac{\partial}{\partial t} f \right].$$

$$= \frac{\partial}{\partial z}\left[\pm V \left(\frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t} \right)\right].$$

$\therefore$ $$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial z}\left[\pm V \frac{\partial f}{\partial z} (\pm V)\right]. \qquad \because \text{From (A) } \frac{\partial z}{\partial t} = \pm V.$$

$$= (\pm V)^2 \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z} \right).$$

$$\frac{\partial^2 y}{\partial t^2} = V^2 \frac{\partial^2 f}{\partial z^2} \quad \text{--- (c)}.$$

Putting the values of $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2}$ from (b) and (c) in ① we get.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 y}{\partial t^2} \quad \text{--- ①}$$

$$\therefore \quad \frac{\partial^2 f}{\partial z^2} = \frac{1}{V^2}\left(V^2 \frac{\partial^2 f}{\partial z^2} \right).$$

$$\therefore \quad \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial z^2}.$$

Since L.H.S = R.H.S. So we arrive at the result from eq. ② i.e,

$y(x, t) = f(x \pm Vt)$ is the solution of wave equation ①.

Power & Intensity in Wave Motion:

When we move the end of string up and down, we give energy to the string. This energy is transmitted through the string to the other end. So wave motion is the mechanism by which energy can be transformed from one place to the other.

So when wave passes along the string then each particle performs work on the particle to its right and work is done on the particle itself by the particle to its

In this way energy is transferred from particle to particle. When the string is moved, the particle of the string have transverse velocity 'u' given as follows.

$$y(x,t) = y_m \sin(kx - \omega t)$$

$$\therefore \quad u(x,t) = \frac{dy}{dt}$$

$$u(x,t) = -\omega y_m \cos(kx - \omega t) \rightarrow (a)$$

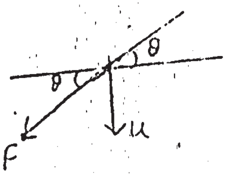
Let F be the force exerted on the element of the string by the element left to it.

Now power transmitted is given by,

$$P = \vec{F}.\vec{u} = Fu = F_y u.$$

This is so because only y-comp of force contributes to the power.

$$\therefore \quad P = u F_y \quad - ①$$

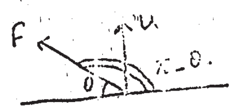


$$\text{Now } F_y = F \sin\theta.$$

Now for small angle $\sin\theta \simeq \tan\theta$.

$$\therefore \quad F_y = F \tan\theta.$$

If we consider the angle with +ve x-axis then



$$F_y = F \tan(\pi - \theta) = -F \tan\theta.$$

$$\therefore \quad F_y = -F\left(\frac{dy}{dx}\right) \quad - (b)$$

Putting (a) and (b) in ① we get

$$P = -\omega y_m \cos(kx - \omega t) \; F \frac{dy}{dx}.$$

$$P = -\omega y_m \cos(kx - \omega t)(-F) \frac{d}{dx}\left[y_m \sin(kx - \omega t)\right].$$

$$P = +\omega y_m \cos(kx - \omega t) \; F \left[k y_m \cos(kx - \omega t)\right].$$

$$P = y_m^2 \omega k F \cos^2(kx - \omega t)$$

$$\text{Because} \quad v = \sqrt{\frac{F}{\mu}} \Rightarrow v^2 = \frac{F}{\mu}$$

$$\text{or} \quad F = \mu v^2.$$

$$\therefore \quad P = y_m^2 \omega k \mu v^2 \cos^2(kx - \omega t). \qquad \because k = \frac{\omega}{v}$$

$$kv = \omega.$$

$$\therefore \quad P = y_m^2 \omega^2 \mu v \cos^2(kx - \omega t)$$

It should be noted that power (rate of flow of energy) is not constant. This is because the inpower oscillates. So we take the average power which is given by,

$$\bar{P} = y_m^2 \mu v \omega^2 \langle \cos^2(kx - \omega t) \rangle.$$

$$\because \langle \cos^2(kx - \omega t) \rangle = \frac{1}{2}.$$

$$\therefore \bar{P} = \frac{1}{2} y_m^2 \mu v \omega^2.$$

This shows that average power does not depend on x and t but it depends on square of amplitude and square of frequency.

In a three dimensional wave e.g light or sound wave, we define intensity of the wave as,

"The average power per unit area placed ⊥ to the direction of propagation of the wave"

Mathematically, $I = \frac{\bar{P}}{A}$

$I \propto A^2$

$A \propto \frac{1}{r}$

$A \propto 1/r^2$.

The intensity of a wave travelling along a string is propotional to the square of its amplitude but it is not true for a circular or spherical wave because amplitude of spherical wave is not constant and is a function of distance from the point source.

The intensity of a spherical wave from a point source located in an isotropic medium is given by,

$$I = \frac{P}{A} = \frac{P}{4\pi r^2} = \frac{P}{4\pi}\left(\frac{1}{r^2}\right)$$

$$I = \text{constt} \times \frac{1}{r^2}.$$

$$\therefore I \propto \frac{1}{r^2}.$$

$$\because I \propto (\text{Amplitude})^2.$$

$$\therefore (\text{Amplitude})^2 \propto \frac{1}{r^2}.$$

$$\therefore \text{Amplitude} \propto \frac{1}{r}.$$

So we find that amplitude of spherical wave is inversely propotional to distance from the point source i.e if the distance from the source is doubled, the amplitude reduces to half and intensity decreases four times.

they are said to interfere and the phenomenon is called interference.

Consider two sinusoidal waves of same amplitude and frequency travelling along x-axis with the same speed. These waves are represented as

$$y_1(x,t) = y_m \sin(kx - \omega t - \varphi_1)$$
$$y_2(x,t) = y_m \sin(kx - \omega t - \varphi_2)$$

Where φ_1 and φ_2 are phase constts for two waves.

The resultant wave is given by the principle of superposition as

$$y = y_1(x,t) + y_2(x,t)$$
$$y = y_m \sin(kx - \omega t - \varphi_1) + y_m \sin(kx - \omega t - \varphi_2)$$
$$= y_m [\sin(kx - \omega t - \varphi_1) + \sin(kx - \omega t - \varphi_2)]$$

By Trigonometry: $\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$

$$\therefore y = y_m \, 2 \sin\left(\frac{kx - \omega t - \varphi_1 + kx - \omega t - \varphi_2}{2}\right) \cos\left(\frac{kx - \omega t - \varphi_1 - kx + \omega t + \varphi_2}{2}\right)$$

$$y = y_m \left[2 \sin\left(\frac{2kx - 2\omega t - \varphi_1 - \varphi_2}{2}\right) \cos\left(\frac{\varphi_2 - \varphi_1}{2}\right)\right]$$

$$y = y_m \left[2 \sin\left(\frac{2kx - 2\omega t}{2} - \frac{\varphi_1 + \varphi_2}{2}\right) \cos\left(\frac{\varphi_2 - \varphi_1}{2}\right)\right]$$

$$y = y_m \left[2 \sin\left((kx - \omega t) - \left(\frac{\varphi_1 + \varphi_2}{2}\right)\right) \cos\left(\frac{\varphi_2 - \varphi_1}{2}\right)\right]$$

$$= 2 y_m \cos\left(\frac{\varphi_2 - \varphi_1}{2}\right) \sin\left(kx - \omega t - \frac{\varphi_1 + \varphi_2}{2}\right)$$

Putting $\frac{\varphi_1 + \varphi_2}{2} = \varphi'$.

and $\varphi_2 - \varphi_1 = \Delta\varphi$.

$$y = 2 y_m \cos\frac{\Delta\varphi}{2} \sin(kx - \omega t - \varphi') \quad \text{--- (1)}$$

From (1) we find that the resultant wave has same frequency but its amplitude is $2y_m \cos\frac{\Delta\varphi}{2}$.

$\therefore$ Amplitude of resultant wave $= 2y_m \cos\frac{\Delta\varphi}{2}$.

Case-I (Constructive Interference):

When $\frac{\Delta\varphi}{2} = 0$ i.e. $\Delta\varphi = 0$.

Then amplitude of resultant wave $= 2y_m$ which is twice the

the magnitude of either wave. In this case crest of one wave falls on the crest of other wave and trough of one wave falls on the trough of the other wave. This is called constructive interference.

So for constructive intercuce

$$\text{Phase difference} = \frac{1}{2}\Delta\varphi = 0, \pi, 2\pi, \ldots$$

$$\text{OR} \quad \frac{1}{2}\Delta\varphi = n\pi.$$

$$\text{OR} \quad \boxed{\Delta\varphi = 2n\pi}$$

Now we derive the condition of constructive interference in terms of path difference.

$$\text{As} \quad \text{Path difference} = (\text{Phase diff}) \times \frac{\lambda}{2\pi}$$

$$\therefore \text{Path diff} = (2n\pi)\frac{\lambda}{2\pi}$$

$$\text{Path diff.} = n\lambda.$$

If path diff is denoted by Δx then

$$\Delta x = n\lambda.$$

$$\text{or} \quad \Delta x = 0, \lambda, 2\lambda, \ldots$$

Case II (Destructive Interference):

When $\frac{\Delta\varphi}{2} = 90°$ or $\Delta\varphi = 180°$.

Then amplitude of resultant wave is $= 2y_m \cos 90° = 0$.

In this case crest of one wave falls on the trough of the other wave and vice versa. This is called destructive interference.

So for destructive interference.

$$\text{Phase diff} = \frac{1}{2}\Delta\varphi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\text{or} \quad \frac{1}{2}\Delta\varphi = (2n+1)\pi/2.$$

$$\text{or} \quad \boxed{\Delta\varphi = (2n+1)\pi.}$$

In terms of path diff. the condition is given as

$$\text{Path diff.} = (\text{Phase difference}) \times \frac{\lambda}{2\pi}$$

$$\text{Path diff.} = (2n+1)\pi \times \frac{\lambda}{2\pi}.$$

Path diff. $(2n+1)\frac{\lambda}{2}$.

or $\Delta x = (2n+1)\frac{\lambda}{2}$.

or $\Delta x = \lambda/2, \frac{3\lambda}{2}, \frac{5\lambda}{2}, \ldots\ldots$

×————×

Sample Problem-4

Two waves travel in the same direction along a string and interfere. The waves have the same amplitude and the same wavelength and travel with the same speed. The amplitude of the waves is 9.7 mm and there is a phase difference of 110° between them. (a) What is the amplitude of the combined waves? To what value should the phase difference be changed so that the combined wave will have an amplitude equal to that one of the original waves?

Sol: $y_m = 9.7\,mm$, $\Delta\varphi = 110°$

(a) Amplitude of combined wave = ?

As amplitude of resultant wave $= 2y_m \cos\frac{1}{2}\Delta\varphi$.

$= 2 \times 9.7 \times \cos\frac{1}{2}(110)$

$= 19.4 \times \cos 55$.

$= 19.4\,(0.573) = 11.1\,mm$.

$\boxed{\text{Amplitude of Combined wave} = 11.1\,mm}$ Ans.

(b) What is $\Delta\varphi$ if resultant $= y_m$.

∴ According to the given condition

$$2y_m \cos\frac{1}{2}\Delta\varphi = y_m.$$

$$2\cos\frac{1}{2}\Delta\varphi = 1.$$

$$\cos\frac{1}{2}\Delta\varphi = \cos^{-1} 1/2.$$

$$\frac{1}{2}\Delta\varphi = 60°$$

$$\therefore \Delta\varphi = 120°.$$ Ans.

×————×

Sample Problem-5.

A listener is seated at a point distant 1.2 m directly in front of one speaker. The two speakers which are separated by a distance of D = 2.3 m emit pure tones of wavelength λ. The waves are in phase when they leave the speakers. For what wavelength will the listener hear a minimum in the sound intensity? [Fig. 2.8]

Sol:- Let $x = 1.2\,m$, $D = 2.3\,m$, $\lambda = ?$ for min. sound.

As for minimum sound the two waves should interfere destructively.

the magnitude of either wave. In this case crest of one wave falls on the crest of other wave and trough of one wave falls on the trough of the other wave. This is called constructive interference.

So for constructive interference

$$\text{Phase difference} = \frac{1}{2}\Delta\varphi = 0, \pi, 2\pi, \ldots$$

$$\text{OR} \quad \frac{1}{2}\Delta\varphi = n\pi.$$

$$\text{OR} \quad \boxed{\Delta\varphi = 2n\pi}$$

Now we derive the condition of constructive interference in terms of path difference.

$$\text{As} \quad \text{Path difference} = (\text{Phase diff}) \times \frac{\lambda}{2\pi}.$$

$$\therefore \text{Path diff} = (2n\pi)\frac{\lambda}{2\pi}$$

$$\text{Path diff.} = n\lambda.$$

If path diff is denoted by Δx then

$$\Delta x = n\lambda.$$

$$\text{or} \quad \Delta x = 0, \lambda, 2\lambda, \ldots$$

Case II (Destructive Interference):

When $\frac{\Delta\varphi}{2} = 90°$ or $\Delta\varphi = 180°$.

Then amplitude of resultant wave is $= 2y_m \cos 90° = 0$.

In this case crest of one wave falls on the trough of the other wave and vice versa. This is called destructive interference.

So for destructive interference.

$$\text{Phase diff} = \frac{1}{2}\Delta\varphi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\text{or} \quad \frac{1}{2}\Delta\varphi = (2n+1)\pi/2.$$

$$\text{or} \quad \boxed{\Delta\varphi = (2n+1)\pi.}$$

In terms of path diff. the condition is given as

$$\text{Path diff.} = (\text{Phase difference}) \times \frac{\lambda}{2\pi}$$

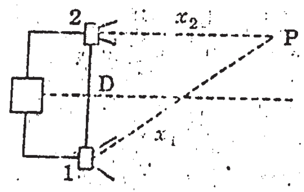
$$\text{Path diff.} = (2n+1)\pi \times \frac{\lambda}{2\pi}.$$

$\therefore$ Path diff $= (n + \frac{1}{2})\lambda$.

or $\Delta x = (n + \frac{1}{2})\lambda$.

$x_1 - x_2 = (n + \frac{1}{2})\lambda$ — (1)

If the listener is seated in front of speaker 2 then



$x_2 = 1.2\,m$

$\therefore x_1^2 = x_2^2 + y^2$

$x_1 = \sqrt{x_2^2 + y^2}$

$= \sqrt{(1.2)^2 + (2.3)^2} = \sqrt{1.44 + 5.29}$

$= \sqrt{6.73} = 2.59$.

$x_1 = 2.6m$

Putting the values in (1) we get

$2.6 - 1.2 = (n + \frac{1}{2})\lambda$.

$1.4 = (n + \frac{1}{2})\lambda$.

$1.4 = \frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2}, \ldots$ (Put $n = 0, 1, 2, 3, \ldots$).

$\therefore \lambda = 2(1.4), \frac{2}{3}(1.4), \frac{2}{5}(1.4), \ldots$

$\boxed{\lambda = 2.8\,m,\ 0.93\,m,\ 0.56\,m, \ldots}$ Ans.

x———x

Standing Waves:

"When two exactly similar waves moving in a medium along the same straight line in opposite directions. Suppose the resultant pattern formed is called standing or stationary wave."

These waves are not actually stationary but only appears to be so. These waves are called standing waves because no moving wave is visible on the string the pattern formed seems to be stationary.

In standing waves there are certain points on the string which do not vibrate and their amplitude of vibration is is zero. Such points are called Nodes. Between two nodes there are points where amplitude of vibration is maximum. Such points are called ANTINODES. So standing waves consists of nodes and antinodes. The distance between two consecutive nodes or antinodes is equal to $\frac{\lambda}{2}$.

Mathematical Analysis of Standing Waves:

Consider two identical waves moving along a string in opposite directions. These waves are represented as

$$y_1(x,t) = y_m \sin(kx - \omega t) \quad \text{towards right}$$

$$y_2(x,t) = y_m \sin(kx + \omega t) \quad \text{towards left.}$$

By the superposition principle, the resultant wave is

$$y(x,t) = y_1(x,t) + y_2(x,t).$$

$$y(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t)$$

$$= y_m [\sin(kx - \omega t) + \sin(kx + \omega t)]$$

From Trigonometry $\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$

$$\therefore \quad y(x,t) = 2y_m \left[\sin\left(\frac{kx - \omega t + kx + \omega t}{2}\right) \cos\left(\frac{kx - \omega t - kx - \omega t}{2}\right)\right]$$

$$= 2y_m [\sin kx \cos(-\omega t)] \qquad \because \cos(-\theta) = \cos\theta.$$

This is the equation of a standing wave. This equation ~~back~~ shows that each particle performs S.H.M with same angular frequency 'ω' but amplitude of different particles are different because amplitude varies with x.

Points of Max. Amplitude:

In eq. ① amplitude is given by

$$A = 2y_m \sin kx \quad — ②$$

We find that A is max $= 2y_m$ when

$$kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \qquad \because k = \frac{2\pi}{\lambda}$$

$$\frac{2\pi}{\lambda}x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\text{or} \quad x = \frac{\pi}{2}\left(\frac{\lambda}{2\pi}\right),\ \frac{3\pi}{2}\left(\frac{\lambda}{2\pi}\right),\ \frac{5\pi}{2}\left(\frac{\lambda}{2\pi}\right), \ldots$$

$$x = \frac{\lambda}{4},\ \frac{3\lambda}{4},\ \frac{5\lambda}{4}, \ldots$$

$$\text{or} \quad x = (2n+1)\frac{\lambda}{4}.$$

These points of max. amplitude are called antinodes.

Points of Minimum Amplitude:

From the expression

$A = 2y_m \sin kx$ we find that amplitude is zero at all points for which

$$kx = 0, \pi, 2\pi, 3\pi, \ldots$$

$$\frac{2\pi}{\lambda}x = 0, \pi, 2\pi, 3\pi, \ldots \quad \because k = \frac{2\pi}{\lambda}$$

$$x = 0,\ \pi\left(\frac{\lambda}{2\pi}\right),\ 2\pi\left(\frac{\lambda}{2\pi}\right),\ 3\pi\left(\frac{\lambda}{2\pi}\right), \ldots$$

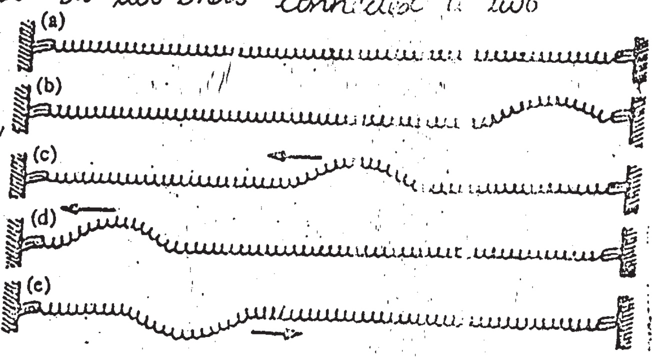
$$x = 0,\ \frac{\lambda}{2},\ 2\left(\frac{\lambda}{2}\right),\ 3\left(\frac{\lambda}{2}\right), \ldots$$

$$x = n\left(\frac{\lambda}{2}\right).$$

These points of zero amplitude are called nodes. The distance b/w a node and an antinode is $\frac{\lambda}{4}$.

Phase Changes on Reflection:

Consider a coil of spring with its two ends connected to two hooks as shown.



Suppose two hooks are denser than the spring.

Suppose we produce an upward displacement in the spring at right end. This upward displacement travels towards left hook and goes on raising the spring. The vibrations of spring are ⊥ to the direction of propagation of the displacement. So it is a Transverse pulse.

On reversing the left end, the upward displacement tries to raise the hook also. But the hook being denser is not raised. Due to the reaction of hook, the upward displacement becomes down displacement.

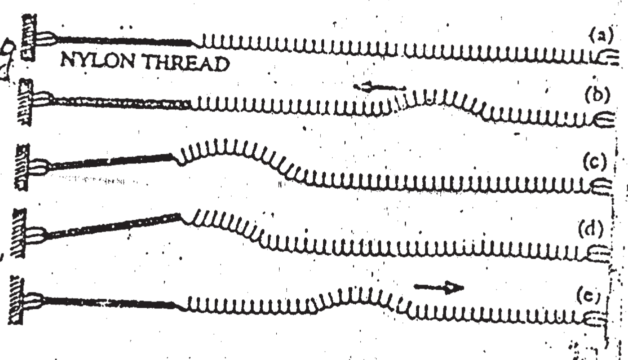
"The bouncing back of a wave into the same medium after striking the boundary of another medium is called reflection.

Thus we have a general rule,

"When a transverse wave is reflected from a denser medium, it is reflected with opposite phase i.e upward displacement is reflected as a downward displacement & vice versa"

The reflected wave has a phase difference of 180° or a path difference of $\frac{\lambda}{2}$.

Now suppose the left end of the spring is tied to a medium rare than the spring e.g nylon thread.



Now an upward displacement produced at right end can raise the nylon thread on reaching it. A part of the pulse travels into the nylon thread and a part goes back as upward displacement.

Thus we find that

"When a transverse wave is reflected at the boundary of a rare medium, the wave is reflected without reversal of i.e upward displacement is reflected as upward displacement and vice versa.

$$\text{or} \quad x = \frac{\pi}{2}\left(\frac{\lambda}{2\pi}\right), \; \frac{3\pi}{2}\left(\frac{\lambda}{2\pi}\right), \; \frac{5\pi}{2}\left(\frac{\lambda}{2\pi}\right), \ldots$$

$$x = \frac{\lambda}{4}, \; \frac{3\lambda}{4}, \; \frac{5\lambda}{4}, \ldots$$

$$\text{or} \quad x = (2n+1)\frac{\lambda}{4}.$$

These points of max. amplitude are called antinodes.

Points of Minimum Amplitude:

From the expression

$A = 2y_m \sin kx$ we find that amplitude is zero at all points for which

$$kx = 0, \pi, 2\pi, 3\pi, \ldots$$

$$\frac{2\pi}{\lambda}x = 0, \pi, 2\pi, 3\pi, \ldots \qquad \because k = \frac{2\pi}{\lambda}$$

$$x = 0, \pi\left(\frac{\lambda}{2\pi}\right), 2\pi\left(\frac{\lambda}{2\pi}\right), 3\pi\left(\frac{\lambda}{2\pi}\right), \ldots$$

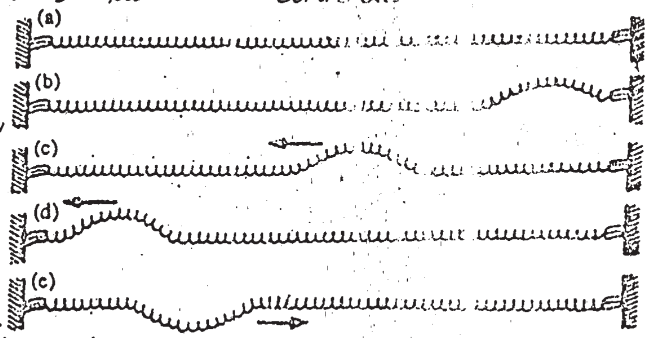
$$x = 0, \frac{\lambda}{2}, 2\left(\frac{\lambda}{2}\right), 3\left(\frac{\lambda}{2}\right), \ldots$$

$$x = n\left(\frac{\lambda}{2}\right).$$

These points of zero amplitude are called nodes. The distance b/w a node and an antinode is $\frac{\lambda}{4}$.

Phase Changes on Reflection

Consider a coil of spring with its two ends connected to two hooks as shown.



Suppose two hooks are denser than the spring.

Suppose we produce an upward displacement in the spring at right end. This upward displacement travels towards left hook and goes on raising the spring. The vibrations of spring are ⊥ to the direction of propagation

The part of the wave which travels into the other medium is called transmitted wave.

The passage of wave from one medium into the other is called transmission of wave. The relative portions of the wave that are reflected and transmitted depend upon inertia and elasticity of the one medium as compared to the other.

During transmission the frequency of wave does not change only the speed and wave length change.

Natural Frequency & Resonance:

When a system (simple pendulum) is made to vibrate, it vibrates with a certain time period and certain frequency. Whenever this pendulum is set vibrating, it will always vibrate with the same period and same frequency called its "natural period" and natural frequency. Natural frequency or natural period depends upon the length of pendulum.

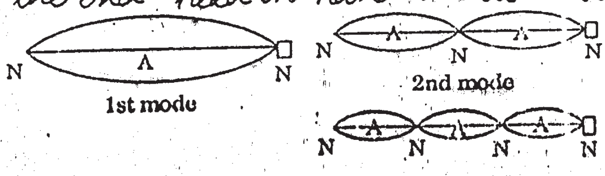
Consider a string of length 'L' stretched with one fixed to a hook and the other end held in hand.

When we move the end held in hand up and down we send transverse waves to the hook. When these waves reach the hook and are reflected back. Due to superposition b/w incident and reflected waves, stationary waves are formed.

If we wiggle the string with different frequencies, the string vibrates in different modes. Fig. shows different modes of oscillation of the string. These modes depend upon frequency.

The end fixed to hook and the end held in hand act as nodes.

The spacing b/w two consecutive modes is $= \lambda/2$.



3rd mode

To produce standing waves in the string the following condition should be satisfied.

$$L = n\frac{\lambda_n}{2} \qquad n = 1, 2, 3, \ldots\ldots$$

For nth mode, the wave length is given by

$$\lambda_n = \frac{2L}{n} \qquad n = 1, 2, 3, \ldots.$$

In terms of frequency, we get. As $v = \nu_n \lambda_n$..

$$\lambda_n = \frac{v}{\nu_n}.$$

$$\therefore \frac{v}{\nu_n} = \frac{2L}{n}.$$

$$\text{or} \quad \frac{\nu_n}{v} = \frac{n}{2L}.$$

$$\boxed{\nu_n = n\left(\frac{v}{2L}\right)} \qquad n = 1, 2, 3, \ldots.$$

So we should shake the string at these particular frequencies to produce the standing waves. These frequencies are called natural frequency.

Resonance: When the frequency of shaking becomes equal to any of natural frequencies, the string begins to vibrate with greater amplitude which we call resonance.

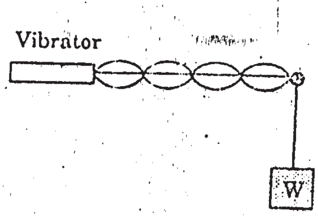
So resonance may be defined as

"An increase in the amplitude of a vibrating body under the action of a periodic force whose frequency is equal to natural frequency of vibrating body".

At resonance the string absorbs as much energy as it can from the source. This is true for any vibrating ~~body~~ system. e.g in tuning a radio, the natural frequency of electronic circuit is changed untill it becomes equal to the particular frequency of radio waves that are broadcast from the station. At this point resonance produces and absorbs as much energy from the signal it can. Similarly resonance can take place in sound, electromagnetism, optics and atomic & nuclear physics.

Sample Problem 6

In the arrangement [Fig. 2.12] a vibrator sets the string in motion at a frequency of 120 HZ. The string has a length L = 1.2 m and its linear mass density is 1.6 gm/cm. To what value must tension be adjusted by increasing the hanging weight to obtain the pattern of motion with four loops.



Sol Frequency $= \nu_4 = 120\,Hz$, $L = 1.2\,m$, $\mu = 1.6\,g/cm$
$= .0016\,g/m$

Tension $= F = ?$

As. $V = \sqrt{\frac{F}{\mu}}$

and $\nu_n = n\left(\frac{V}{2L}\right)$

$$\nu_n = \frac{n}{2L}\left(\sqrt{\frac{F}{\mu}}\right)$$

$$\nu_n^2 = \frac{n^2}{4L^2}\left(\frac{F}{\mu}\right)$$

$$F = \frac{4L^2\mu^2\nu_n}{n^2} \quad \text{——} \ (1)$$

As there are four loops $\therefore n = 4$.

$$\therefore F = \frac{4L^2\mu\nu_4^2}{4^2}$$

$$= \frac{L^2\mu\nu_4^2}{4}$$

$$= \frac{(1.2)^2 \times (.0016) \times (120)^2}{4}$$

$F = 8.3\,N$ Ans

Sample Problem-7

A violin string tuned to concert A(440 Hz) has a length 0.234 m. (a) What are the three longest wavelengths of the resonances of the string. (b) What are the corresponding wavelengths that reach the ear of the listener.

Sol:- $\nu = 440\,Hz$; $L = 0.34\,m$.

(a) Three longest wave lengths of resonance = ?

As $L = n\frac{\lambda_n}{2}$

or $\lambda_n = \frac{2L}{n}$

Putting $n = 1, 2, 3$, we get

$$\lambda_1 = \frac{2L}{1} = 2 \times .34 = 0.68m.$$

$$\boxed{\lambda_1 = 0.68m} \text{ Ans}$$

$$\lambda_2 = \frac{2L}{2} = L = 0.34m$$

$$\boxed{\lambda_2 = 0.34m} \text{ Ans.}$$

$$\lambda_3 = \frac{2L}{3} = \frac{2 \times 0.34}{3} = 0.23m$$

$$\boxed{\lambda_3 = 0.23m} \text{ Ans}$$

(b) Corresponding wave lengths that reach the ear = ?

As we know that when a wave passes from one medium into another, both speed and wave length change but frequency remains unchanged.

For lowest frequency, speed is given by

$$V_1 = \nu \lambda_1.$$

$$= 440 \times 0.68.$$

$$V_1 = 299 m/s = V_{string}$$

Wave speed in air is = 343 m/s (To learn).

$$\therefore \nu_{string} = \nu_{air}$$

$$\frac{V_{string}}{\lambda_{string}} = \frac{V_{air}}{\lambda_{air}} \qquad \text{But } \nu = \frac{V}{\lambda}.$$

$$\therefore \lambda_{air} = \frac{\lambda_{string}}{V_{string}} \times V_{air}.$$

$$\lambda_{air} = \lambda_{string} \times \frac{V_{air}}{V_{string}}$$

$$= \lambda_{string} \times \frac{343}{299.}$$

$$\lambda_{air} = \lambda_{string} \times 1.15$$

$\therefore$ Wave lengths in air corresponding to different modes are given as,

$$\lambda_{air} = \lambda_{string} \times 1.15.$$

$$\lambda_1 = 0.68 \times 1.15$$

$\boxed{\lambda_1 = 0.78m}$ Ans for 1^{st} mode.

$\lambda_2 = 0.34 \times 1.15$

$\boxed{\lambda_2 = 0.39m}$ Ans for II^{nd} mode.

$\lambda_3 = 0.23 \times 1.15$

$\boxed{\lambda_3 = 0.26m}$ for III^{rd} mode.