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SPECIAL THEORY OF RELATIVITY

1. Trouble With Classical Mechanics:

The Kinematics developed by Galileo and the mechanics developed by Newton are the basis of classical physics.

The classical physics has explained the motion of planets successfully. It has explained the properties of gases on the basis of kinetic theory.

However a number of experimental phenomenon of Photo-electric effect, Radioactivity, Black body radiation etc. could not be explained on the basis of classical physics.

A few troubles faced by classical physics are as under:

(i) **Troubles with Ideas about Time**. Pion (π^+ or $-$) is a particle that can be created in a high energy particle accelerator. It is a very unstable particle. Pions created at rest have an average half life of 26 ns while pions created in motion at a speed of 0.913c have an average half life of 63.7 ns. This average half life is much larger than the average half life measured for pions at rest. This effect is called time dilation.

This shows that relative motion b/w pion & the laboratory has dilated the time factor by 2.5. The effect of time dilation can't be explained by Classical physics. According to Classical physics time has same values for all observers.

(ii) **Troubles with Ideas about Length** an observer in the laboratory at rest measure the distance b/w the location of pions formation & of its decay. Suppose he measures the distance as 17.4 m.

Now consider another observer moving with the pion at a speed of 0.913c. To this observer pion appears to be at rest. He measures life time of pion to be 26n.s. According to this observer the distance b/w the two locations is

$$S = Vt = 0.913c \times 26 \times 10^{-9} = 7.1m.$$

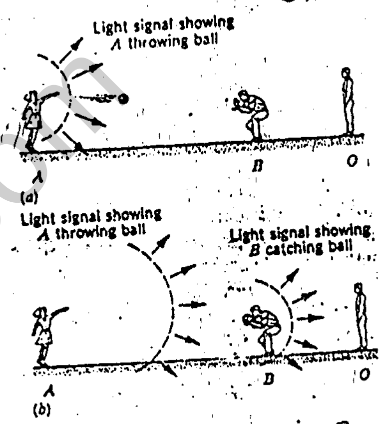
Thus the two observers in relative motion measure different values for the same length interval.

This result can't be explained by classical Physics according to which co-ordinates have same values for all observers.

(iii) Troubles with Ideas about Velocity:

Consider three observers A, B and O as shown in figs.

Figs



All the observers are at rest.

Suppose observer 'A' throws a ball with a speed $v > c$ towards observer B who catches it.

The light signal showing "A throwing ball" travels to observer 'O' as does the light signal showing 'B' catching the ball. Both light signals travel at speed c which is less than the speed of ball thrown by A.

So the light signal from 'B' reaches 'O' before the light signal from A. So observer 'O' concludes that B catches the ball before A throws it.

According to classical physics, the bodies can be accelerated to unlimited velocities. However the above trouble about velocity shows that no material object can exceed the speed of light.

2. Postulates of Special Theory of Relativity:

"Theorey of relativity shows the effects of relative motion on physical quantities."

These effects are observed at relativistic speed.

"The speed $\geq C/10$ is called relativistic speed."

where C is the speed of light.

Einstein announced his special theorey of relativity in 1905.

He based his special theorey of relativity on the following two basic postulates.

Statements: (i) The Principle of Relativity

"The laws of Physics have the same form in all inertial frames."

(ii) The Principle of Constancy of Speed of Light

The speed of light in free space has the same value 'C' in all inertial frames of reference.

Discussion:

The first postulate shows that laws of physics are absolute and universal and are same for all inertial observers. So the laws of Physics that hold for one inertial observer can't be violated for anyother inertial observer.

To understand IInd postulate, consider three observers A, B and C at rest in three different inertial frames.

A flash of light emitted by observer 'A' is observed by him to travel at speed 'C'.

If the frame of B is moving away from 'A' at a speed of $C/4$ then according to Galilean Kinematics, B measures the value

$c - \frac{c}{4} = \frac{3c}{4}$ for the speed of flash emitted by A

If the frame of (C) is moving towards A with a speed of $c/4$ then according to Galilean Kinematics, C measures the value of $c + \frac{c}{4} = \frac{5c}{4}$ for the speed of flash emitted by A

However according to IInd postulate all the three observers measure the same speed of flash of light.

However ordinary objects do not obey IInd postulate. e.g

Velocity of a projectile fired from a moving car w.r.t ground is = velocity of projectile w.r.t. car + velocity of car w.r.t. ground.

But velocities of waves and particles moving at speeds close to 'c' do not behave in this way.

When Einstein put forward these postulates, there was no experimental test for the verification of these postulates.

However in 1964 a photon accelerator produced a beam of neutral pions (π^0) which rapidly decayed into γ rays.

$$\pi^0 = \gamma + \gamma$$

Now γ rays are electromagnetic waves and move with the speed of light.

The speed of moving Pions was measured equal to 0.99995C.

According to Galileo, the γ-ray emitted in the direction of motion of pions should have a speed equal to C + .99995C. But the measured speed of γ-rays was equal to C. This is consistent with IInd postulate.

Hence the result of both two postulates is that no material particle can gain the speed equal to or greater than c, no matter how much K.E is given to it.

Classical physics puts no upper limit on the speed of object but special theory of relativity does impose the upper limit. This is the difference b/w classical physics and postulates of special theory of of relativity.

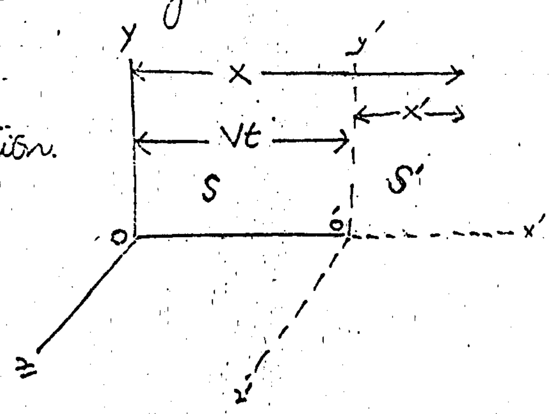
In another experiment in 1964, electrons were the accelerated by a potential difference upto 1.5 MV and the speed of electron was measured. It was found that no matter, how much the accelerating voltage is increased, the speed never reaches or exceeds c. which in consistent with the postulates of special theory of relativity.

3. The Lorentz Transformation:

"The Lorentz transformation is a set of observations made by two observers in different frames of reference which prepare the constancy of speed of light in all system."

Consider two observers in two different inertial frames S and S′. Frame 'S' is at rest and S′ is moving with uniform velocity V along x-axis w.r.t. frame S. Suppose at t=0, the origins of two frames coincide.

Both the observers observe the same event. The position and time of event observed by S is denoted by (x, y, z, t) and position and time of event observed by S′ is denoted by (x′, y′, z′, t′)

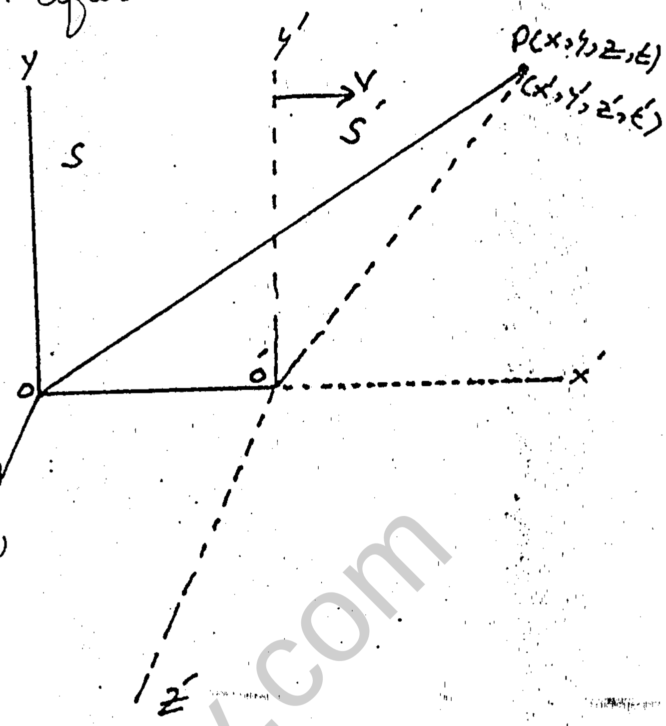


According to Galilean transformation.

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \quad (A)$$

Let a wave of light starts from O and O′ at $t=0$ with speed 'C'.

Let the wave reaches a point P after time t.



Then $OP = Ct$ = Distance covered by light in time 't' w.r.t. S and

$O'P = Ct'$ = Distance covered by light in time t' w.r.t. S′.

So according to 'S'

$$x^2+y^2+z^2 = c^2t^2 \quad \text{— (a)}$$

$$x'^2+y'^2+z'^2 = c^2t'^2 \quad \text{— (b)}$$

From (a) and (b)

$$x^2+y^2+z^2-c^2t^2=0$$

$$x'^2+y'^2+z'^2-c^2t'^2=0.$$

Hence comparing these equations we get,

$$\boxed{x^2+y^2+z^2-c^2t^2 = x'^2+y'^2+z'^2-c^2t'^2} \quad \text{— ①}$$

This is the fundamental eq. of special theory relativity given by Einstein in 1905.

Putting the values of x', y', z', t' from Ⓐ in ① we get,

$$x^2+y^2+z^2-c^2t^2 = (x-vt)^2+y^2+z^2-c^2t^2.$$

$$x^2 = (x-vt)^2$$ which is clearly impossible untill $t=0$.

Hence Galilean transformations fail to satisfy the fundamental equation of special theory of relativity.

Hence we need such transformations which satisfy eq. ①.

Such transformations are called Lorentz transformations.

These are given below,

$$
\left.
\begin{aligned}
x' &= \frac{x - Vt}{\sqrt{1 - V^2/c^2}} = \gamma(x - Vt) \\
y' &= y. \\
z' &= z. \\
t' &= \frac{t - xV/c^2}{\sqrt{1 - V^2/c^2}} = \gamma\left(t - \frac{xV}{c^2}\right)
\end{aligned}
\right\} \text{— (B)}
$$

The set of equations (B) is called Lorentz transformation.
These transformations can be derived as follows.

Derivation of Lorentz Transformation:

As relative velocity V is along x-axis and x-axis ~~and~~ x′-axis coincide. So

$$y' = y \quad \text{and} \quad z' = z$$

So fundamental equation of special theory of relativity becomes,

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2.$$

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \quad \text{— (C)}$$

As O′ appears to move along x-axis with relative velocity V w.r.t. 'S'. So distance covered by O′ w.r.t. S = Vt.
As O also appears to move along x′-axis with relative velocity −V. So distance covered by O w.r.t. S′ = −Vt′.
These two requirements can be satisfied by putting

$$x' = a(x - Vt). \quad \text{— (a)}$$

$$\text{and } x = b(x' + Vt') \quad \text{— (b)}.$$

In equations (a) and (b) if we know 'a' and 'b' then we can find the relations b/w (x, y, z, t) and (x', y', z', t') which satisfy equation (1) (A) which is fundamental eqn of the of relativity.
To find 'a' and 'b' we put the value of x′ from

eq. (a) in eq. (b) we get,

$$x = b[a(x - vt) + vt']$$
$$x = ab(x - vt) + bvt'$$
$$x = abx - abvt + bvt'$$
$$x - abx + abvt = bvt'$$

or $$bvt' = x - abx + abvt$$
$$t' = \frac{x}{bv} - \frac{ax}{v} + at$$
$$t' = at - \frac{ax}{v} + \frac{x}{bv}$$
$$t' = at - \frac{ax}{v}\left(1 - \frac{1}{ab}\right)$$
$$t' = a\left[t - \frac{x}{v}\left(1 - \frac{1}{ab}\right)\right] \quad \text{--- (d)}$$

Putting the values of x' and t' from (a) and (d) in (c) we get

$$x^2 - c^2t^2 = x'^2 - c^2t'^2 \quad \text{--- (c)}$$
$$x^2 - c^2t^2 = [a(x - vt)]^2 - c^2a^2\left[t - \frac{x}{v}\left(1 - \frac{1}{ab}\right)\right]^2$$
$$x^2 - c^2t^2 = a^2(x - vt)^2 - c^2a^2\left[t - \frac{x}{v}\left(1 - \frac{1}{ab}\right)\right]^2$$
$$x^2 - c^2t^2 = a^2(x^2 + v^2t^2 - 2xvt) - c^2a^2\left[t^2 - \frac{2xt}{v}\left(1 - \frac{1}{ab}\right) + \frac{x^2}{v^2}\left(1 - \frac{1}{ab}\right)^2\right]$$
$$x^2 - c^2t^2 = a^2x^2 + a^2v^2t^2 - 2a^2xvt - c^2a^2t^2 + \frac{2c^2a^2xt}{v}\left(1 - \frac{1}{ab}\right) - \frac{c^2a^2x^2}{v^2}\left(1 - \frac{1}{ab}\right)^2$$
$$x^2 - a^2x^2 + \frac{c^2a^2x^2}{v^2}\left(1 - \frac{1}{ab}\right)^2 + 2a^2xvt - \frac{2c^2a^2xt}{v}\left(1 - \frac{1}{ab}\right) - c^2t^2 - a^2v^2t^2 + c^2a^2t^2 = 0$$
$$x^2\left[1 - a^2 + \frac{c^2a^2}{v^2}\left(1 - \frac{1}{ab}\right)^2\right] + xt\left[2a^2v - \frac{2a^2c^2}{v}\left(1 - \frac{1}{ab}\right)\right] + t^2\left[-c^2 - a^2v^2 + c^2a^2\right] = 0$$

This relation must hold for all values of x and t. So the coefficients of x^2, xt and t^2 must be zero seperately.

So we get three equations

$$1 - a^2 + \frac{c^2a^2}{v^2}\left(1 - \frac{1}{ab}\right)^2 = 0 \quad \text{--- (e)}$$

$$2a^2v - \frac{2c^2a^2}{v}\left(1 - \frac{1}{ab}\right) = 0 \quad \text{--- (f)}$$

$$-c^2 + c^2a^2 - a^2v^2 = 0 \quad \text{--- (g)}$$

From (g).

$$c^2 = c^2a^2 - a^2v^2$$

$$c^2 = a^2(c^2 - v^2)$$

$$c^2 = a^2c^2(1 - v^2/c^2)$$

$$1 = a^2(1 - v^2/c^2)$$

or $$a^2 = \frac{1}{(1 - v^2/c^2)}$$

$$\boxed{a = \frac{1}{\sqrt{1 - v^2/c^2}}} \quad \text{--- (h)}$$

From eq. (f) we get,

$$2a^2v = \frac{2c^2a^2}{v}\left(1 - \frac{1}{ab}\right)$$

or $$v^2 = c^2\left(1 - \frac{1}{ab}\right)$$

$$v^2/c^2 = \left(1 - \frac{1}{ab}\right)$$

or $$\frac{1}{ab} = \left(1 - \frac{v^2}{c^2}\right)$$

$$\therefore \frac{1}{ab} = \frac{1}{a^2}$$

From eq. 'h'

$$a^2 = \frac{1}{1 - v^2/c^2}$$

$$\therefore \frac{1}{a^2} = \left(1 - \frac{v^2}{c^2}\right)$$

$$\frac{1}{b} = \frac{1}{a}$$

or $a = b$.

$\therefore$ Putting $a = b$ in eq. 'h' we get.

$$b = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \text{--- (i)}$$

Putting the values of (a) and (b) from (h) and (i) in equations (a) and (b) we get.

$$x' = a(x - vt) \quad \text{—— (a)}$$

$$x = b(x' + vt') \quad \text{—— (b)}$$

$$\therefore \quad x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}$$

$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$

where $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ is called Lorentz factor.

Putting $a = b = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ in eq (d) we get,

$$t' = a\left[t - \frac{x}{v}\left(1 - 1/ab\right)\right] \quad \text{—— (d)}$$

$$t' = \frac{1}{\sqrt{1 - v^2/c^2}}\left[t - \frac{x}{v}\left\{1 - \frac{1}{\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \cdot \frac{1}{\sqrt{1 - v^2/c^2}}}\right\}\right]$$

$$t' = \frac{1}{\sqrt{1 - v^2/c^2}}\left[t - \frac{x}{v}\left\{1 - \frac{1}{\frac{1}{1 - v^2/c^2}}\right\}\right]$$

$$t' = \frac{1}{\sqrt{1 - v^2/c^2}}\left[t - \frac{x}{v}\left\{1 - (1 - v^2/c^2)\right\}\right]$$

$$t' = \frac{1}{\sqrt{1 - v^2/c^2}}\left[t - \frac{x}{v}\left(1 - 1 + v^2/c^2\right)\right]$$

$$= \frac{1}{\sqrt{1 - v^2/c^2}}\left[t - \frac{x}{v}\left(\frac{v^2}{c^2}\right)\right]$$

$$t' = \frac{1}{\sqrt{1 - v^2/c^2}}\left[t - \frac{xv}{c^2}\right]$$

$$t' = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} \qquad \because \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma$$

$$\therefore \quad \boxed{t' = \gamma\left(t - \frac{xv}{c^2}\right)}$$

Hence Lorentz Transformation eq.s become,

$$\left\{ \begin{aligned} x' &= \gamma(x - vt) = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{xv}{c^2}\right) = \frac{t - xv/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned} \right\}$$

This set of equations is called Lorentz Transformations. When $v << c$ i.e $v/c \approx 0$, the Lorentz Transformations transform into Galilean Transformations.

Thus Galilean Transformation is a special case of Lorentz transformations.

Inverse Lorentz Transformation:

Question:

From the Lorentz Transformation.

$$x' = \gamma(x - vt) \quad \text{--- (1)}$$

$$y' = y \quad \text{--- (2)}$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{xv}{c^2}\right) \quad \text{--- (3)}$$

Find the inverse Lorentz Transformations.

Solution: As the Lorentz transformations are

$$x' = \gamma(x - vt) \quad \text{--- (1)}$$

$$y' = y \quad \text{--- (2)}$$

$$z' = z \quad \text{--- (3)}$$

$$t' = \gamma(t - xv/c^2) \quad \text{--- (4)}$$

From eq. (4)

$$\frac{t'}{\gamma} = t - \frac{xv}{c^2}$$

$$\therefore \quad t'/\gamma + \frac{xv}{c^2} = t.$$

Putting the value of 't' in eq (1) we get.

$$x' = \gamma(x - vt) \quad \text{--- (1)}$$

$$x' = \gamma\left[x - v\left(\frac{t'}{\gamma} + \frac{xv}{c^2}\right)\right]$$

$$\frac{x'}{\gamma} = x - \frac{vt'}{\gamma} - \frac{xv^2}{c^2}$$

$$x'/\gamma = x - \frac{xv^2}{c^2} - \frac{vt'}{\gamma}$$

$$\frac{x'}{\gamma} = x(1 - v^2/c^2) - \frac{vt'}{\gamma}$$

$$x\left(1 - \frac{v^2}{c^2}\right) = \frac{x'}{\gamma} + \frac{vt'}{\gamma} = \frac{1}{\gamma}(x' + vt') \longrightarrow (5)$$

As we know that

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\therefore \quad \gamma^2 = \frac{1}{1 - v^2/c^2}$$

$$\frac{1}{\gamma^2} = (1 - v^2/c^2)$$

$\therefore$ Eq. (5) becomes

$$x\left(\frac{1}{\gamma^2}\right) = \frac{1}{\gamma}(x' + vt') = \frac{\gamma^2}{\gamma}(x' + vt')$$

$$\boxed{\therefore \quad x = \gamma(x' + vt')} \quad \text{--- (A)}$$

From eq. (1)

$$\frac{x'}{\gamma} = x - vt.$$

$$\frac{x'}{\gamma} + vt = x.$$

Putting this value of 'x' in eq. (4) we get,

$$t' = \gamma\left(t - \frac{xv}{c^2}\right) \quad — (4)$$

$$t' = \gamma\left[t - \left(\frac{x'}{\gamma} + vt\right)\frac{v}{c^2}\right]$$

$$\frac{t'}{\gamma} = t - \frac{x'v}{c^2\gamma} - \frac{v^2 t}{c^2}$$

OR $$t = \frac{t'}{\gamma} + \frac{x'v}{\gamma c^2} + \frac{v^2 t}{c^2}$$

$$t - \frac{v^2 t}{c^2} = \frac{t'}{\gamma} + \frac{x'v}{\gamma c^2}$$

$$t\left(1 - \frac{v^2}{c^2}\right) = \frac{1}{\gamma}\left(t' + \frac{x'v}{c^2}\right)$$

As we know that $1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}$

$$t\left(\frac{1}{\gamma^2}\right) = \frac{1}{\gamma}\left(t' + \frac{x'v}{c^2}\right) \text{ or } t = \frac{\gamma^2}{\gamma}\left(t' + \frac{x'v}{c^2}\right)$$

$$\boxed{t = \gamma\left(t' + \frac{x'v}{c^2}\right)} \quad — (B)$$

Hence the inverse Lorentz transformations are

$$x = \gamma(x' + vt')$$

$$y = y'$$

$$z = z'$$

$$t = \gamma\left(t' + \frac{x'v}{c^2}\right)$$

where $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$ is called Lorentz factor.

Note.

We can obtain the inverse Lorentz transformations just by interchang primed and unprimed co-ordinates and replacing v by $-v$.

Sample Problem-1

Sample Problem 1 Muons are elementary particles with a (proper) lifetime of 2.2 μs. They are produced with very high speeds in the upper atmosphere when cosmic rays (high-energy particles from space) collide with air molecules. Take the height L_0 of the atmosphere (its rest length) to be 100 km in the reference frame of the Earth, and find the minimum speed that will enable the muons to survive the journey to the surface of the Earth. Solve this problem in two ways: (a) in the Earth's frame of reference and (b) in the muon's frame of reference.

Sol.: $t_0 = 2.2\,\mu S = 2.2\times10^{-6} S.$

$l_0 = 100\,Km = 100\times10^3 = 10^5 m.$

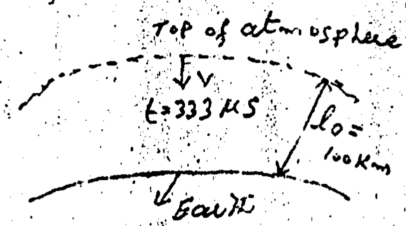
(a) $V=?$ in the earth frame of reference.

(b) $V=?$ in the moon frame of reference.

(a) In the Earth's frame of reference;

If the speed of moon is $\simeq c$.

Then time taken by the moon to travel from top of atmosphere to Earth is



$$t = \frac{l_0}{c} = \frac{10^5}{3\times10^8} = \frac{1}{3}\times10^{-3} = .333\times10^{-3}$$

$$t = 333\times10^{-6} S.$$

$$\boxed{\therefore t = 333\,\mu S}$$

$\therefore$ Speed of moon is given by.

$$t = \frac{t_0}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$t^2 = \frac{t_0^2}{1-\frac{V^2}{c^2}} \qquad 1-\frac{V^2}{c^2} = \frac{t_0^2}{t^2}$$

$$1-\frac{t_0^2}{t^2} = \frac{V^2}{c^2}$$

$$V^2 = c^2\left(1-\frac{t_0^2}{t^2}\right)$$

$$V^2 = \left(1-\frac{(2.2\times10^{-6})^2}{(333\times10^{-6})^2}\right)c^2$$

$$V^2 = \left(1-\frac{(2.2)^2}{(333)^2}\right)c^2$$

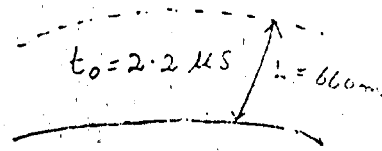
$$= \left(\frac{(333)^2-(2.2)^2}{(333)^2}\right)c^2$$

$$V^2 = \left(\frac{110889-4.84}{110889}\right)c^2$$

$$v^2 = .9999563c^2$$

$$\boxed{\therefore v = 0.999972\,c} \quad \text{Ans.}$$

(b) In the moon's frame of reference, the entire atmosphere is moving with a speed 'c'.



$\therefore$ Height of atmosphere is

$$l = ct_0.$$

$$= 3\times10^8 \times 2.2\times10^{-6} = 6.6\times10^2.$$

$$l = 660m.$$

$\therefore$ Speed of moon is given by.

$$l = l_0\sqrt{1-\frac{v^2}{c^2}}$$

$$l^2 = l_0^2\left(1-\frac{v^2}{c^2}\right).$$

$$\frac{l^2}{l_0^2} = 1-\frac{v^2}{c^2}$$

$$\frac{v^2}{c^2} = 1 - \frac{l^2}{l_0^2}.$$

$$v^2 = \left(1-\frac{l^2}{l_0^2}\right)c^2.$$

$$= \left(1 - \frac{(660)^2}{(10^5)^2}\right)c^2$$

$1 - 4.35\times10^{5}$

$$= \left(1 - \frac{435600}{10^{10}}\right)c^2$$

$$= \left(\frac{10^{10} - 435600}{10^{10}}\right)c^2$$

$$v^2 = .9999564\,c^2$$

$$\boxed{v = .999978c} \quad \text{Ans.}$$

The Transformation of Velocities:

We shall use the equations of Lorentz transformations to get a relation b/w velocity 'V' of a particle measured by an observer in S frame and velocity v' of the same particle measured by an observer in S' frame who is moving with velocity u w.r.t. S.

Suppose according to 'S' particle moves from (x_1, y_1, z_1, t_1) to (x_2, y_2, z_2, t_2) and according S', the particle moves from (x_1', y_1', z_1', t_1') to (x_2', y_2', z_2', t_2')

Let us calculate $V_x' = \frac{\Delta x'}{\Delta t'}$, the x-component of velocity in S'.

By Lorentz transformation, we have

$$x' = \gamma(x - ut)$$

$$t' = \gamma\left(t - u\frac{x}{c^2}\right)$$

$$\Delta x' = \gamma(\Delta x - u\Delta t)$$

$$\Delta t' = \gamma\left(\Delta t - u\frac{\Delta x}{c^2}\right)$$

$$\therefore V_x' = \frac{\Delta x'}{\Delta t'}$$

$$V_x' = \frac{\gamma(\Delta x - u\Delta t)}{\gamma\left(\Delta t - u\frac{\Delta x}{c^2}\right)}$$

$$V_x' = \frac{\Delta x - u\Delta t}{\Delta t - u\frac{\Delta x}{c^2}} = \frac{\Delta t\left(\frac{\Delta x}{\Delta t} - u\right)}{\Delta t\left(1 - \frac{u\,\Delta x}{\Delta t\,c^2}\right)}$$

$$\boxed{\therefore V_x' = \frac{V_x - u}{1 - u V_x/c^2}} \quad \text{--- (1)} \qquad \because \frac{\Delta x}{\Delta t} = V_x$$

$$y' = y$$

$$\Delta y' = \Delta y$$

$$\frac{\Delta y'}{\Delta t'} = \frac{\Delta y}{\Delta t'}$$

$$\frac{\Delta y'}{\Delta t'} = \frac{\Delta y}{\gamma\left(\Delta t - \frac{u\Delta x}{c^2}\right)}$$

$$V_y' = \frac{\Delta y}{\gamma\Delta t\left(1 - \frac{u}{c^2}\frac{\Delta x}{\Delta t}\right)}$$

$$V_y' = \frac{\Delta y/\Delta t}{\gamma\left(1 - \frac{uV_x}{c^2}\right)}$$

$$V_y' = \frac{V_y}{\gamma\left(1 - \frac{uV_x}{c^2}\right)}$$

Similarly $V_z' = \frac{V_z}{\gamma\left(1 - \frac{uV_x}{c^2}\right)}$

Similarly for y and z components, we have

$$\left.\begin{aligned} V_y' &= \frac{V_y}{\gamma\left(1 - \frac{uV_x}{c^2}\right)} \\ \text{and} \quad V_z' &= \frac{V_z}{\gamma\left(1 - \frac{uV_x}{c^2}\right)} \end{aligned}\right\} \quad \text{--- (2)}$$

It should be noted that $V_y \neq V_y'$ and $V_z \neq V_z'$ even though $\Delta y = \Delta y'$ and $\Delta z = \Delta z'$. This is the difference b/w Lorentz and Galilean transformations. Eq. (1) and (2) give Lorentz velocity transformations.

Inverse Velocity Transformations:

We can obtain inverse velocity transformations from eq.s of velocity transformations simply by co-ordinates changing u by $-u$ & primed & unprimed co-ordinates.

So inverse Lorentz velocity transformations

$$\left.\begin{aligned} V_x &= \frac{V'_x + u}{\left(1 + \frac{u V'_x}{c^2}\right)} \\ V_y &= \frac{V'_y}{\gamma\left(1 + \frac{u V'_x}{c^2}\right)} \\ V_z &= \frac{V'_z}{\gamma\left(1 + \frac{u V'_x}{c^2}\right)} \end{aligned}\right\} \quad \text{--- (A)}$$

The set of equations (A) is called inverse Lorentz velocity transformations.

Lorentz Velocity Transformations (under non-relativistic limits)

Now we examine equations (1) and (2) under non-relativistic limits i.e $u \ll c$. then $\frac{u}{c^2} = 0$.

$\therefore$ Equations (1) and (2) become

$$\left.\begin{aligned} V'_x &= V_x - u \\ V'_y &= V_y \\ V'_z &= V_z \end{aligned}\right\}$$

which are Galilean transformations.

So under non-relativistic limits, Lorentz velocity transformations change into Galilean transformations.

The Lorentz Velocity Transformation & Einstein 2nd Postulate:

We can derive the result of Einstein's II postulate from

Lorentz-velocity transformations. According to Einstein's II postulate, the speed of light is constt for all observers. So speed 'c' measured by an observer must also be measured to be 'c' by any other observer.
Suppose the two observers observe a common event of passage of light beam along x-axis in frame S and S'.
According to observers in S, the velocity of light become along x-axis $V_x = c$ and $V_y = V_z = 0$.
From eq. ① and ②, the velocity of light beam measured by S' is

$$V'_x = \frac{V_x - u}{1 - uV_x/c^2} = \frac{c-u}{1 - \frac{uc}{c^2}} = \frac{c-u}{1-\frac{u}{c}}$$

$$V'_x = \frac{c-u}{\frac{c-u}{c}} = c$$

$$\boxed{V'_x = c \quad \text{and} \quad V'_y = V'_z = 0}$$

So the observer in S' also measures the same speed.
Thus the speed of light is same for all observers.

Sample Problem.3.

Sample Problem 3 In inertial frame S, a red light and a blue light are separated by a distance $\Delta x = 2.45$ km, with the red light at the larger value of x. The blue light flashes, and 5.35 μs later the red light flashes. Frame S' is moving in the direction of increasing x with a speed of $u = 0.855c$. What is the distance between the two flashes and the time between them as measured in S'?

Sol: $\Delta x = 2.45$ km $= 2.45 \times 10^3$ m, $\Delta x' = ?$
$\Delta t = 5.35\,\mu s = 5.35 \times 10^{-6}$ s ; $\Delta t' = ?$
$u = 0.855c$.

Using Lorentz transformation.

$x' = \gamma(x - ut)$ — ① and $t' = \gamma\left(t - \frac{ux}{c^2}\right)$ — ②

Now $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \dfrac{1}{\sqrt{1-(.855)^2}}$

$\gamma = \dfrac{1}{\sqrt{1-0.731025}} = \dfrac{1}{\sqrt{0.268975}}$

$= \dfrac{1}{0.518628}$

$\boxed{\gamma = 1.928}$

From eq. ① $\Delta x' = \gamma(\Delta x - u\Delta t)$.

Putting the values of γ, Δx, u and Δt.

$\Delta x' = 1.928\,(2.45\times10^3 - 0.855c \times 5.35\times10^{-6})$.

$\Delta x' = 1.928\,(2450 - 0.855\times3\times10^8\times5.35\times10^{-6})$.

$= 1.928\,(2450 - 25.5\times535.35)$

$= 1.928\,(2450 - 1372.275)$

$= 1.928\,(1077.725) = 2077.8\text{ m}$

$\boxed{\Delta x' = 2.077\text{ km}}$ Ans.

From eq ② $\Delta t' = \gamma\left(\Delta t - \frac{u\Delta x}{c^2}\right)$.

Putting the values of γ, Δt, u, Δx, c we get.

$\Delta t' = 1.928\left(5.35\times10^{-6} - \dfrac{.855\times3\times10^8\times2.45\times10^3}{(3\times10^8)^2}\right)$

$= 1.928\left(5.35\times10^{-6} - \dfrac{.855\times2.45\times10^3}{3\times10^8}\right)$

$= 1.928\,(5.35\times10^{-6} - 0.69825\times10^{-5})$

$= 1.928\,(5.35\times10^{-6} - 6.9825\times10^{-6})$

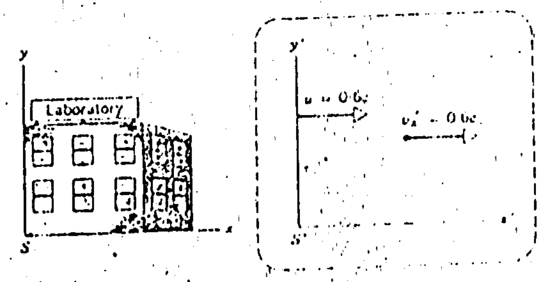
$= 1.928\,(-1.6325)\times10^{-6}$

$= -3.14746\times10^{-6} = -3.15\times10^{-6}$ sec.

$\boxed{\Delta t' = -3.15\,\mu\text{Sec}}$ Ans:

Sample Problem 4

Sample Problem 4 A particle is accelerated from rest in the laboratory until its velocity is 0.60c. As viewed from a frame that is moving with the particle at a speed of 0.60c relative to the laboratory, the particle is then given an additional increment of velocity amounting to 0.60c. Find the final velocity of the particle as measured in the laboratory frame.



Sol:- The lab frame is $= S$.

The particle frame is $= S'$.

∴ The velocity of S' w.r.t $S = u = .60c$.

Velocity of particle w.r.t. $S' = V_x' = .60c$.

Final velocity of particle w.r.t. $S = V_x = ?$

By using inverse Lorentz velocity transformations, we have

$$V_x = \frac{V_x' + u}{1 + uV_x'/c^2} = \frac{.60c + .60c}{1 + \frac{(.60c)(.60c)}{c^2}}$$

$$V_x = \frac{1.20c}{1 + .36} = \frac{1.20c}{1.36} = 0.88c$$

$\boxed{V_x = .88c}$ Ans

Consequences of Lorentz Transformation

We shell discuss the following consequences of Lorentz Transformation:

① Relativity of Time ② Relativity of Length.

③ Relativity of Mass.

(i) Relativity of Time (Time Dilation)

Consider two frames of reference S and S'. S is at rest and S' is moving with uniform velocity u w.r.t. S.

Suppose an event occurs at one and the same place 'x' in frame S. The duration of this event measured by the observer in frame S is $\Delta t = t_2 - t_1$.

The duration of the same event measured by the observer in frame S' is $\Delta t' = t_2' - t_1'$.

By Using Lorentz transformations, we get

$$t_2' = \gamma \left(t_2 - \frac{x_2 u}{c^2}\right).$$

and $$t_1' = \gamma \left(t_1 - \frac{x_1 u}{c^2}\right)$$

$$\therefore \Delta t' = t_2' - t_1'$$

$$= \gamma \left(t_2 - \frac{x_2 u}{c^2}\right) - \gamma \left(t_1 - \frac{x_1 u}{c^2}\right)$$

$$= \gamma \left[t_2 - \frac{x_2 u}{c^2} - t_1 + \frac{x_1 u}{c^2}\right]$$

$$= \gamma \left[(t_2 - t_1) - \frac{u}{c^2}(x_2 - x_1)\right]$$

because event occurs at the same place

$$\therefore x_1 = x_2.$$

$$\therefore x_2 - x_1 = 0.$$

$$\therefore \Delta t' = \gamma [(t_2 - t_1)]$$

$$\therefore \gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

$$\Delta t' = \frac{t_2 - t_1}{\sqrt{1 - u^2/c^2}} = \frac{\Delta t}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Since $\frac{1}{\sqrt{1 - u^2/c^2}} > 1.$

So $\boxed{\Delta t' > \Delta t}$

Hence the observer in S' will conclude that the clock in frame S is slowed down i.e time is lengthened i.e dilated.

This effect is called time dilation.

(ii) Relativity of Length (Length Contraction)

Consider a rod lying along x-axis in a stationary frame S. The rod is at rest in frame S.

Let the co-ordinates of its ends be x_1 and x_2.

Then $\quad l = x_2 - x_1 \quad$ —(1).

Let the length of the rod seen in the moving frame S' be l'.

The co-ordinates of ends of the rod are x_1' and x_2'.

Then $\quad l' = x_2' - x_1' \quad$ —(2)

It should be noted that the measurements are made simultaneously in both the frames.

By using inverse Lorentz transformations we get

$$x = \gamma(x' + ut')$$

$$\therefore x_2 = \gamma(x_2' + ut_2')$$

$$x_1 = \gamma(x_1' + ut_1')$$

$$\therefore x_2 - x_1 = \gamma[x_2' - x_1' + u(t_2' - t_1')] \qquad \because \Delta t' = t_2' - t_1'$$

$$x_2 - x_1 = \gamma[x_2' - x_1' + u\Delta t']$$

Putting $\Delta t' = 0$ because measurements are made simultaneously

$$\therefore x_2 - x_1 = \gamma(x_2' - x_1')$$

But $x_2 - x_1 = l$

and $x_2' - x_1' = l'$.

$$\therefore l = \gamma l'.$$

$$l' = \frac{1}{\gamma} l. \qquad \because \frac{1}{\gamma} = \sqrt{1 - u^2/c^2}$$

$$l' = l\sqrt{1 - \frac{u^2}{c^2}}$$

Since $\sqrt{1 - \frac{u^2}{c^2}} < 1.$

$\therefore \quad l' < l$

i.e the length l of the rod appears to be reduced in the moving frame S'. The length of the rod measured in the frame in which it is at rest is called proper length. This effect is called Lorentz contraction.

(iii) **Relativity of Mass (Prove that $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$**

$$As \quad E = mc^2 .$$

$$\therefore \quad dE = d(mc^2)$$

$$dE = c^2 dm \quad \text{---} ①$$

$$\text{Also} \quad dE = F dx \qquad \because F = \frac{dP}{dt}$$

$$dE = \frac{dP}{dt} dx . \qquad P = mv$$

$$dE = \frac{d}{dt}(mv)\, dx$$

$$dE = \left(m\frac{dv}{dt} + v\frac{dm}{dt} \right) dx$$

Putting $V = \frac{dx}{dt}$.

$$\text{or} \quad dx = v dt .$$

$$\therefore dE = \left(m\frac{dv}{dt} + v\frac{dm}{dt} \right) v dt .$$

$$dE = mv dv + v^2 dm - ② .$$

Comparing ① and ② we get.

$$c^2 dm = mv dv + v^2 dm .$$

$$c^2 dm - v^2 dm = mv dv .$$

$$(c^2 - v^2)\, dm = mv dv .$$

$$\frac{dm}{m} = \frac{v dv}{c^2 - v^2} = \frac{v dv}{c^2\left(1 - \frac{v^2}{c^2}\right)}$$

$$dm = \frac{\frac{v}{c^2} dv .}{\left(1 - \frac{v^2}{c^2}\right)} \qquad \because d\left(1 - \frac{v^2}{c^2}\right) = \frac{-2v}{c^2} dv$$

$$\therefore \quad \frac{dm}{m} = -\frac{\frac{1}{2}\left(\frac{2V\,dV}{c^2}\right)}{1-\frac{V^2}{c^2}}$$

Integrating both sides we get,

$$\int_{m_0}^{m} \frac{dm}{m} = -\frac{1}{2}\int_{0}^{V} \frac{-\frac{2V}{c^2}\,dV}{1-V^2/c^2}$$

$$\Big|\ln m\Big|_{m_0}^{m} = \left[-\frac{1}{2}\ln\left(1-\frac{V^2}{c^2}\right)\right]_0^V$$

$$\ln m - \ln m_0 = \ln\left(1-\frac{V^2}{c^2}\right)^{-1/2}$$

$$\ln\frac{m}{m_0} = \ln\left(1-\frac{V^2}{c^2}\right)^{-1/2}$$

$$\frac{m}{m_0} = \left(1-\frac{V^2}{c^2}\right)^{-1/2}$$

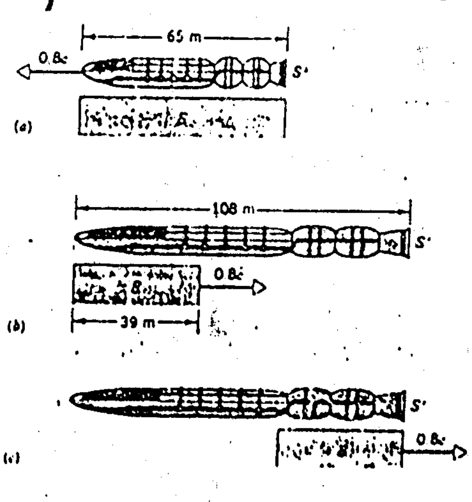
$$\frac{m}{m_0} = \frac{1}{\left(1-V^2/c^2\right)^{1/2}}$$

$$\boxed{m = \frac{m_0}{\sqrt{1-V^2/c^2}}}$$

This is the relativistic formula for the variation of mass with velocity and is called relativistic mass.

Sample Problem 5

[redacted] (An observer S is standing on a platform of length $D_0 = 65$ m on a space station. A rocket passes at a relative speed of $0.80c$ moving parallel to the edge of the platform. The observer S notes that the front and back of the rocket simultaneously line up with the ends of the platform at a particular instant (Fig. 18*a*). (*a*) According to S, what is the time necessary for the rocket to pass a particular point on the platform? (*b*) What is the rest length, L_0 of the rocket? (*c*) According to an observer S' on the rocket, what is the length D of the platform? (*d*) According to S', how long does it take for observer S to pass the entire length of the rocket? (*e*) According to S, the ends of the rocket simultaneously line up with the ends of the platform. Are these events simultaneous to S'?)



Sol: Rest length of plateform $= D_0 = 65m$.

Relative speed of rocket $= u = 0.80c$.

(a) Acc. to 'S' what is the time necessary for the rocket to pass a particular point on the plateform.

Time for rocket to pass a particular point w.r.t S is given by,

$$t_0 = \frac{D_0}{u} = \frac{65}{0.8c} = \frac{65}{0.8 \times 3 \times 10^8}$$

$$t_0 = \frac{65}{2.4} \times 10^{-8} = 27.08 \times 10^{-8} = 0.27 \times 10^{-6}\,\text{sec.}$$

$$\boxed{t_0 = 0.27\,\mu\text{sec}}$$ Ans.

(b) What is the rest length L_0 of the rocket?

As $l = l_0\sqrt{1 - u^2/c^2}$

$$l_0 = \frac{l}{\sqrt{1 - u^2/c^2}} = \frac{65}{\sqrt{1 - \frac{(0.8c)^2}{c^2}}}$$

$$l_0 = \frac{65}{\sqrt{1 - .64}} = \frac{65}{\sqrt{0.36}} = \frac{65}{0.6}$$

$$\boxed{l_0 = 108\,m.}$$ Ans.

(c) What is length D of plateform w.r.t. S'.

$$D = D_0\sqrt{1 - u^2/c^2} = 15\sqrt{1 - \frac{(0.8c)^2}{c^2}}$$

$$= 65\sqrt{1 - 0.64} = 65\sqrt{0.36} = 65 \times .6$$

$$\boxed{D = 39\,m}$$ Ans.

(d) How long does it take for observer 'S' to pass the entire length of the rocket w.r.t. S'?

According to S', the observer S should move a distance equal to rest length of rocket i.e 108 m.

Time in given by,

$$t' = \frac{L_o}{u} = \frac{108}{0.8c} = \frac{108}{0.8 \times 3 \times 10^8}$$

$$t' = \frac{108}{2.4} \times 10^{-8}$$

$$t' = 45 \times 10^{-8} = 0.45 \times 10^{-6} \text{ Sec.}$$

$$\boxed{t' = 0.45 \ \mu \text{Sec}} \text{ Ans.}$$

(e) According to 'S' the ends of the rocket simultaneously line up with the ends of plate form. Are these events simultaneously to S' ?

Sol: Acc. to S', The rocket's length is $= L_o = 108m$ and acc. to S', the contracted length of plateform is $= D = 39m$.

So there is no way for S' to observe the two ends of the rocket to align up simultaneously. So two events that are simultaneous to S' can't be simultaneous to S'.

5. Equivalence Of Mass And Energy.

i.e Prove that $E = mc^2$.

Suppose a force F acts on a body and the body covers a distance dx. The work done by the force is given by,

$$dw = \vec{F} \cdot \vec{dx}$$

$$dw = F dx \qquad \because \theta = 0°$$

By work energy theorem, the work done become the increase in K.E of the body.

$$\therefore \ dw = dK$$

$$\therefore \ dK = F dx \quad — (1)$$

In Classical physics force is written as,

$$F = \frac{dP}{dt}$$

$$F = \frac{d\,mv}{dt}$$

$$dK = V dm V + mv dv \qquad \because \frac{dx}{dt} = V.$$

$$dK = V^2 dm + mv dv.$$

$$dK = mv dv + V^2 dm \quad — ②$$

From relativistic mechanics ; $m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$

$$m = \frac{m_0}{(1 - \frac{v^2}{c^2})^{1/2}}$$

$$m = m_0 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$\therefore \quad dm = m_0 \left(-\frac{1}{2}\right) \left(1 - \frac{v^2}{c^2}\right)^{-3/2} \left(-\frac{2v}{c^2}\right) dv.$$

$$dm = \frac{m_0}{(1 - \frac{v^2}{c^2})^{3/2}} \frac{v}{c^2} dv$$

$$dm = \frac{m_0}{(1 - v^2/c^2)^{1/2}} \cdot \frac{1}{(1 - v^2/c^2)} \frac{v}{c^2} dv$$

$$\therefore \quad dm = m \frac{1}{(1 - \frac{v^2}{c^2})} \cdot \frac{v dv}{c^2} \qquad \because \frac{m_0}{\sqrt{1 - v^2/c^2}} = m.$$

$$= \frac{mv\, dv}{\frac{c^2 - v^2}{c^2}} \cdot \frac{1}{c^2}$$

$$dm = \frac{mv\, dv}{c^2 - v^2}$$

$$\text{or} \quad mv\, dv = (c^2 - v^2) dm.$$

$\therefore$ Eq. ② becomes

$$dK = (c^2 - v^2) dm + v^2 dm.$$

$$dK = (c^2 - v^2 + v^2) dm.$$

$$dK = c^2 dm.$$

Integration given $\quad K = c^2 m + A \quad —③$

where A is constt of integration.

At $t = 0$, $m = m_0$ and $K = 0$ $\quad \therefore \quad A = -c^2 m.$

$\therefore$ Eq. ③ becomes,

$$K = c^2 m - c^2 = 0$$

$$K = mc^2 - m_o c^2$$

$$\text{or} \quad mc^2 = K + m_o c^2 \quad — (4)$$

This shows that when $K=0$, the body still possesses some energy $= m_o c^2$ called rest mass energy $= E_o$.

$$\therefore \quad mc^2 = K + E_o$$

But $K + E_o$ = Total energy $= E$.

$$\therefore \quad \boxed{E = mc^2}$$ Hence the proof.

This expression is called Einstein mass energy relation.

6. Relativistic Energy

From the relation $E = mc^2$.

$$E = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$E^2 = \frac{m_o^2 c^4}{1 - \frac{v^2}{c^2}}$$

$$\frac{E^2}{c^2} = \frac{m_o^2 c^2}{1 - \frac{v^2}{c^2}} \quad — (1)$$

Similarly

$$P = mv.$$

$$P = \frac{m_o v}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$P^2 = \frac{m_o^2 v^2}{1 - \frac{v^2}{c^2}} \quad — (2)$$

Subtracting (2) from (1) we get,

$$\frac{E^2}{c^2} - P^2 = \frac{m_o^2 c^2}{1 - v^2/c^2} - \frac{m_o^2 v^2}{1 - v^2/c^2}$$

$$\frac{E^2}{c^2} - P^2 = \frac{m_o^2 c^2 - m_o^2 v^2}{1 - v^2/c^2}$$

This is the expression for relativistic energy E.

Sample Problem-6.

Sample Problem 6 What is the momentum of a proton moving at a speed of $v = 0.86c$?

Sol:- m = rest mass of proton $= 1.67 \times 10^{-27}$ kg.

$P = ?$, $V = 0.86C$.

As $$P = \frac{mV}{\sqrt{1 - V^2/c^2}}$$

$$= \frac{1.67 \times 10^{-27} \times 0.86C}{\sqrt{1 - \frac{(0.86c)^2}{c}}}$$

$$= \frac{1.4362C \times 10^{-27}}{\sqrt{1 - (0.86)^2}} = \frac{1.4362C \times 10^{-27}}{0.51}$$

$$= 2.8 \times 10^{-27} C$$

$$= 2.816 \times 3 \times 10^{8} \times 10^{-27}$$

$$\boxed{P = 8.44 \times 10^{-19} \text{ kg m/s}}$$ Ans

Sample Problem - 7

Sample Problem 7 In the Stanford Linear Collider* electrons are accelerated to a kinetic energy of 50 GeV. Find the speed of such an electron as (a) a fraction of c and (b) a difference from c. The rest energy of the electron is 0.511 MeV = 0.511×10^{-3} GeV.

Sol:- Kinetic energy = K = 50 GeV = $50 \times 10^{9} \times 1.6 \times 10^{-19}$ J = 80×10^{-10} J.

(a) Speed of electron = V = ? (as a fraction of C)

(b) Difference (C - V) = ?

(a) As $K = E - E_0$.

$$= mc^2 - m_0c^2.$$

$$K = \frac{m_0c^2}{\left(1-\frac{V^2}{c^2}\right)^{1/2}} - m_0c^2.$$

$$K = m_0c^2\left(\frac{1}{\left(1-\frac{V^2}{c^2}\right)^{1/2}} - 1\right)$$

$$\frac{K}{m_0c^2} = \frac{1}{\left(1-\frac{V^2}{c^2}\right)^{1/2}} - 1.$$

$$1 + \frac{K}{m_0c^2} = \frac{1}{\left(1-\frac{V^2}{c^2}\right)^{1/2}}.$$

$$\left(1 + \frac{K}{m_0c^2}\right)^2 = \frac{1}{\left(1-V^2/c^2\right)}$$

$$1 - \frac{V^2}{c^2} = \frac{1}{\left(1 + K/m_0c^2\right)^2}.$$

$$1 - \frac{1}{\left(1+\left(K/m_0c^2\right)^2\right)^2} = \frac{V^2}{C^2}.$$

$$\sqrt{1 - \frac{1}{\left(1+\frac{K}{m_0c^2}\right)^2}} = \frac{V}{C} \quad \text{—— (1)}$$

Put $\frac{-1}{\left(1+\frac{K}{m_0c^2}\right)^2} = x$

$$\therefore \frac{V}{C} = (1+x)^{1/2} \quad (2)$$

$$\because K >> m_o c^2$$

$$\therefore x << 1.$$

Using Binomial theorem on (2) we get,

$$\frac{V}{C} = 1 + \frac{1}{2}x.$$

$$= 1 + \frac{1}{2}\left[\frac{-1}{(1+\frac{K}{m_o c^2})^2}\right]$$

$$\frac{V}{C} = 1 - \frac{1}{2(1+\frac{K}{m_o c^2})^2} \quad (3).$$

Now $m_o c^2 = 9.1 \times 10^{-31} \times 9 \times 10^{16}$.

$m_o c^2 = 81.9 \times 10^{-15} J$.

$\therefore$ Equation (3) becomes,

$$\frac{V}{C} = 1 - \frac{1}{2\left(1 + \frac{80 \times 10^{-10}}{81.9 \times 10^{-15}}\right)^2}$$

$$= 1 - \frac{1}{2\left(1 + \frac{80}{81.9} \times 10^5\right)^2}$$

$$\frac{V}{C} = 1 - 0.5\left[\frac{1}{(97681.098)^2}\right]$$

$$= 1 - 0.5\left[\frac{1}{9.54 \times 10^9}\right]$$

$$= 1 - 0.5\left[\frac{1}{9.54} \times 10^{-9}\right]$$

$$= 1 - 0.5[0.105 \times 10^{-9}].$$

$$V/C = 1 - 0.052 \times 10^{-9} = 1 - 5.2 \times 10^{-11}$$

$$\boxed{V = C(1 - 5.2 \times 10^{-11})}$$

(b) $C - V = C - [C(1 - 5.2 \times 10^{-11})]$.

$= C - C(1 - 5.2 \times 10^{-11})$

$= C\,[1 - 1 + 5.2 \times 10^{-11}]$

$= C\,[5.2 \times 10^{-11}] = 5.2 \times 10^{-11} C.$

$= 5.2 \times 10^{-11} \times 3 \times 10^{8}.$

$= 5.2 \times 3 \times 10^{-3}$

$= 15.6 \times 10^{-3} = 0.0156.$

$\boxed{C - v = 0.016\,m/s.}$ Ans.