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Chapter- 38

INDUCTANCE

1- Inductance:

"When a variable current passes through an inductor the magnetic flux passing through the inductor changes and an emf is produced in the inductor. This type of emf is called self induced emf and the phenomenon is called self induction."

The magnitude of induced emf is propotional to the rate change of current.

i.e $e_L \propto \frac{di}{dt}$

$$e_L = L\frac{di}{dt}$$

Where the constant 'L' is called inductance of the coil.

If $\frac{di}{dt} = 1\,amp/sec$ then $e_L = L$.

So inductance of a coil may be defined as,

"emf induced in a coil for unit rate of change of current in the coil."

Unit: The S.I unit of inductance is called Henry.

It is given as

$$L = \frac{e_L}{di/dt}$$

$L = 1\,A$ if $\frac{di}{dt} = 1\,amp/sec$ and $e_L = 1\,volt.$

i.e $1\,H = 1\,Volt \times sec/Amp.$

Definition:

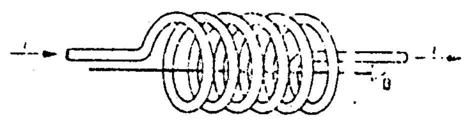
"The inductance of a coil in 1 Henry if current changing at the rate of 1amp/sec, induces an emf of 1 volt in the coil."

The behaviour of self inductance in electrical circuits is similar to inertia in matter. i.e it tends to maintain the current when it is increasing and it tends to maintain the current when it is decreasing.

This relationship between sign of e_L and sign of $\frac{di}{dt}$ is given as the basis of Lenz's law according to which,

The direction of induced emf is always s.t it opposes its own cause.

Fig. shows an ideal conductor in which steady current is flowing. Let us suddenly decrease the current i.



This decrease in current is the change which is opposed by the inductance. So to oppose the decreasing current, the induced emf provides an additional current in the same direction as i as shown in fig. 2a.

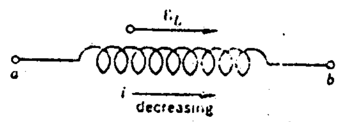


Fig 2a

If we suddenly increase the current, then this increase in current is opposed by the inductance. So to oppose the increasing current the induced emf provides an additional current in a direction opposite to i as shown in fig. 2b.
So in each case the induced emf opposes the change in current.
From fig (2) gives the summary of relationship b/w the sign of $\frac{di}{dt}$ and sign of e_L.

In fig (2a) $V_b > V_a$.

$$\therefore \; V_b - V_a = \left| L \frac{di}{dt} \right|.$$

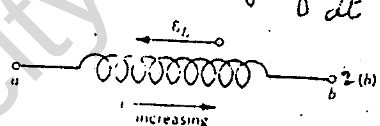


Fig 2b.

Since i is decreasing, so $\frac{di}{dt}$ is $-ve$. So the above relation becomes

$$V_b - V_a = -L \frac{di}{dt}.$$

In fig (2b) $V_b < V_a$. So $V_b - V_a$ is $-ve$. Since i is increasing so $\frac{di}{dt}$ is $+ve$

$$\therefore \; -(V_b - V_a) = L \frac{di}{dt}.$$

$$\therefore \; V_b - V_a = -L \frac{di}{dt}.$$

So the same above equation is obtained in this case also.

2. Calculation of Inductance:

Consider a coil having N turns. Let ϕ_B be the magnetic flux passing through each turn of the coil. Then the product $N\phi_B$ is called number of flux linkages of the inductor.
Now emf induced in the coil is obtained by Faraday's law as

$$e_L = -\frac{d}{dt} N\phi_B \quad -(1).$$

But this induced emf is also given as

$$e_L = -L \frac{di}{dt} \quad -(2).$$

Comparing (1) and (2) we get

$$L \frac{di}{dt} = \frac{d}{dt}(N\phi_B)$$

By integrating we get

$$L i = N \phi_B.$$

$$L = \frac{N \phi_B}{i} \quad -(3)$$

This equation (3) enables us to find inductance of coil.
In eq. (3) ϕ_B is propotional to the current i. So inductance is independent of current i.

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3. The Inductance of Solenoid

Consider a long solenoid having length 'l'. Suppose n is the number of turns per unit length of the solenoid. If i is the current in the solenoid. If i is the current in the solenoid then the magnetic induction due to solenoid is

$$B = \mu_0 n i \quad -(1).$$

If N is the total number of turns. Then number of flux linkages in length 'l' is

$$N\phi_B = (nl)BA.$$

Putting B from (1) we get

$$N\phi_B = (nl)\mu_0 n i A.$$

$$N\phi_B = \mu_0 n^2 l i A. \quad -(2)$$

If L is the inductance of the solenoid then

$$L = \frac{N\phi_B}{i}.$$

Putting value of $N\phi_B$ from (2) we get,

$$L = \frac{\mu_0 n^2 l i A}{i}.$$

$$L = \mu_0 n^2 l A \quad -(3)$$

This equation (3) gives the inductance of long solenoid.
The inductance per unit length of the solenoid is

$$\frac{L}{l} = \mu_0 n^2 A \quad -(4)$$

From eq. (4) we see that inductance per unit length depends on geometry. i.e cross sectional area and number of turns per unit length. Equations (3) and (4) are valid for a solenoid of length greater than its radius. In these equations we have neglected the magnetic lines near the end of solenoid turns, as we neglected the fringing field near the edges of the plates of a capacitor.

The Inductance of Toroid:

Consider a toroid of radius r and total number of turns N on it. If i is the current flowing in it, then the flux density B due it is given by;

$$B = \frac{\mu_o N i}{2\pi r} \quad -(1).$$

This equation shows that B is not constant inside the toroid but varies with radius r.

The magnetic flux through the cross section of toroid is given by

$$\phi_B = \int \vec{B} \cdot d\vec{A}$$

$$\phi_B = \int B dA$$

Where dA is the area of rectangular strip of the toroid shown in fig.

Now area dA of the strip is $= h dr$.

$$\therefore \phi_B = \int_a^b B h dr$$

Putting B from (1) we get

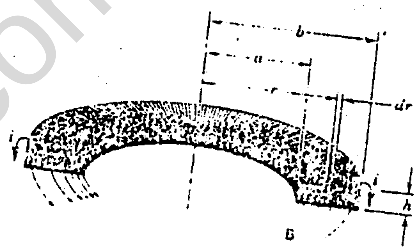
$$\phi_B = \int_a^b \frac{\mu_o N i}{2\pi r} h dr.$$

$$= \frac{\mu_o N i h}{2\pi} \int_a^b \frac{dr}{r}$$

$$= \frac{\mu_o N i h}{2\pi} \left| \ln r \right|_a^b.$$

$$= \frac{\mu_o N i h}{2\pi} \left| \ln b - \ln a \right|$$

$$\phi_B = \frac{\mu_o N i h}{2\pi} \ln \frac{b}{a}$$



If L is the inductance of the coil then

$$L = \frac{N \phi_B}{i}$$

$$= \frac{N}{i} \frac{\mu_o N i h}{2\pi} \ln \frac{b}{a}.$$

$$L = \frac{\mu_o N^2 h}{2\pi} \ln \frac{b}{a}$$

This is the inductance of a toroid.

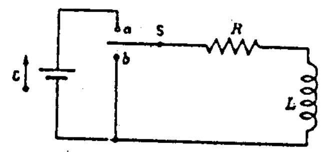
It is clear that inductance depends on geometry of the toroid.

Growth of Circuits in an L.R. Circuit

L.R. Circuit:

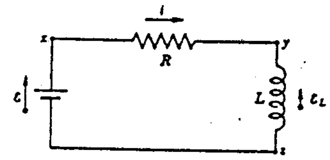
An L.R circuit consists of an inductor L and resistance R connected in series with a battery of emf E.

Fig shows an L.R circuit in series with a battery of voltage E.



Let us see what happen when we close the switch S at 'a' keep open at 'b'.

Fig. A-II shows L.R circuits with switch S closed at 'a'.



When switch S is closed at 'a' the current in R starts increasing. If inductor L was not present, the current will rise rapidly to maximum value $\frac{E}{R}$. But due to inductor an induced emf e_L is produced across 'L'. which opposes this rise in current. So e_L opposes emf E of the battery. Thus the current in R depends upon the sum of two emfs i.e constant emf E of battery and variable emf e_L of inductor. As long as e_L is present, the current in the resistor is less than $\frac{E}{R}$.

By the time, the current increases less rapidly and e_L which is propotional to $\frac{di}{dt}$ becomes smaller. So the current in the circuit approaches $\frac{E}{R}$ exponentially as explained below.

Let us study the circuit quantitatively

By using Kirchhoff's 2nd rule and starting from point x clockwise.

Point x is at a higher potential than y.

$\therefore$ When we traverse R from x to y the change in potential is

$$V_y - V_x = -iR$$

$$\therefore e_R = -iR$$

Point y is at a higher potential than z. Therefore when we traverse inductor L from y to z the change in potential is

$$V_z - V_y = -L\frac{di}{dt}$$

$$\therefore e_L = -L\frac{di}{dt} \qquad \text{where } e_L = V_z - V_y.$$

$\therefore$ Kirchhoff's 2nd rule gives

$$-iR - L\frac{di}{dt} + E = 0$$

$$E = L\frac{di}{dt} + iR \quad -(1)$$

$$L\frac{di}{dt} = E - iR.$$

$$L\frac{di}{dt} = R\left(\frac{E}{R} - i\right)$$

$$\frac{di}{dt} = \frac{R}{L}\left(\frac{E}{R} - i\right)$$

$$\frac{di}{\left(\frac{E}{R} - i\right)} = \frac{R}{L}\,dt$$

$$\text{or} \quad \frac{-di}{\left(\frac{E}{R} - i\right)} = -\frac{R}{L}\,dt.$$

Integration gives

$$\ln\left|\frac{E}{R} - i\right| = -\frac{R}{L}t + A \quad \text{--- (2)}$$

To find the constant 'A' we use initial conditions. i.e at $t=0, i=0 \therefore A = \ln\frac{E}{R}$

$\therefore$ Equation (2) becomes

$$\ln\left(\frac{E}{R} - i\right) = -\frac{R}{L}t + \ln\frac{E}{R}.$$

$$\ln\left(\frac{E}{R} - i\right) - \ln\frac{E}{R} = -\frac{R}{L}t.$$

$$\ln\left(\frac{E/R - i}{E/R}\right) = -\frac{R}{L}t.$$

$$\text{or} \quad \frac{E/R - i}{E/R} = e^{-\frac{R}{L}t}.$$

$$\frac{E}{R} - i = \frac{E}{R}e^{-R/L\,t}.$$

$$i = \frac{E}{R} - \frac{E}{R}e^{-\frac{Rt}{L}}$$

$$i = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right) \quad \text{--- (3)}$$

This equation shows that for $t = \infty$, $i = \frac{E}{R} = i_0$ (say) where i_0 is the maximum value of current.

$$\therefore i = i_0\left(1 - e^{-tR/L}\right) \quad \text{--- (A).}$$

The equation (A) gives the growth of current in the circuit i.e it shows how the current increases with the passage of time.
This equation shows that the current i goes on increasing with time and ultimately attains maximum value. i.e after a long time. From this equation we see that current increases exponentially.

Time Constant τ_L

The ratio $\frac{L}{R} = \tau_L$ is called inductive time constant.

$\therefore$ Equation (3) becomes

$$i = \frac{E}{R}\left(1 - e^{-t/\tau_L}\right) \quad \text{--- (4)}$$

Determination of T_L (i.e Prove that $T_L = \frac{L}{R}$)

T_L can be determined by putting the value of i and $\frac{di}{dt}$ in eq (1)

Diff. eq (4) w.r.t time

$$\frac{di}{dt} = \frac{E}{R}\left[0 - \left(-\frac{1}{T_c}\right)e^{-t/T_c}\right]$$

$$\frac{di}{dt} = \frac{E}{RT_c} e^{-t/T_c} \quad -(5)$$

Putting the value of i and $\frac{di}{dt}$ from (4) and (5) in (1) we get.

$$L\frac{di}{dt} + iR = E \quad -(1).$$

$$\therefore L\frac{E}{RT_L} e^{-t/T_c} + \frac{E}{R}\left(1 - e^{-\frac{t}{T_L}}\right)R = E.$$

$$\frac{L}{RT_L} e^{-\frac{t}{T_c}} + \left(1 - e^{-t/T_L}\right) = 1.$$

$$\frac{L}{RT_L} e^{-\frac{t}{T_L}} \cancel{+1} - e^{-\frac{t}{T_L}} = \cancel{1}.$$

$$\frac{L}{RT_L} e^{-\frac{t}{T_L}} - e^{-\frac{t}{T_L}} = 0.$$

$$\cancel{e^{-t/T_L}}\frac{L}{RT_L} = \cancel{e^{-t/T_L}}$$

$$\frac{L}{R} = T_L.$$

$$\boxed{T_L = \frac{L}{R}}$$ Hence the proof.

Here T_L is called the inductive time constant. It tells how rapidly the current in an LR circuit reaches the maximum value $\frac{E}{R}$.

Show that $T_L = \frac{L}{R}$ has the dimensions of Time:

The T_L has the dimensions of time as proved below

$$|T_L| = \left|\frac{L}{R}\right| = \left|\frac{\text{Henry}}{\text{Ohm}}\right|$$

$$= \frac{\text{Volt - sec}}{\text{Amp} \times \text{ohm}}$$

$$= \frac{\text{Amp - ohm . sec}}{\text{Amp - ohm}}$$

$$E = L\frac{di}{dt}.$$

$$L = E\frac{dt}{di}$$

$$\frac{V}{I} = R$$

$$\frac{\text{Volt}}{\text{Amp}} = \text{ohm}$$

$\boxed{\text{So dimensions of } T_L = \text{sec}}$

Physical Significance of 'T_L' = ?

To define T_L we put $t = T_L$ in eq (4)

$$\therefore i = i_0\left(1 - e^{-\frac{t}{T_L}}\right)$$

$$i = i_0\left(1 - e^{-T_L/T_L}\right)$$

$$= i_0\left(1 - e^{-1}\right) = i_0\left(1 - \frac{1}{e}\right)$$

$$= i_0\left(1 - 1/2.718\right) = i_0\,(0.63)$$

$$\boxed{i = 0.63 i_0}$$

∴ "Time constant is the time after which the current becomes 0.63 of the maximum current."

It is clear that growth of current depends on T_L. The smaller the value of T_L, the more rapid the growth of charge. As $T_L = \frac{L}{R}$. Thus greater the resistance R, the smaller will be the time constant.

Fig (i) shows the growth of current i with time t.

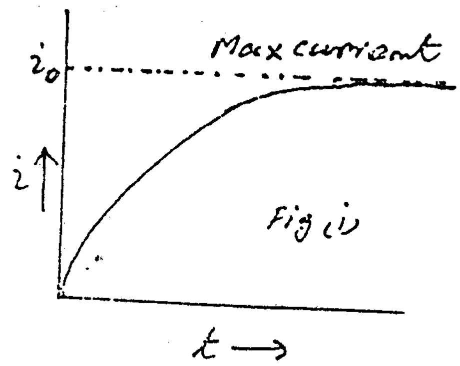
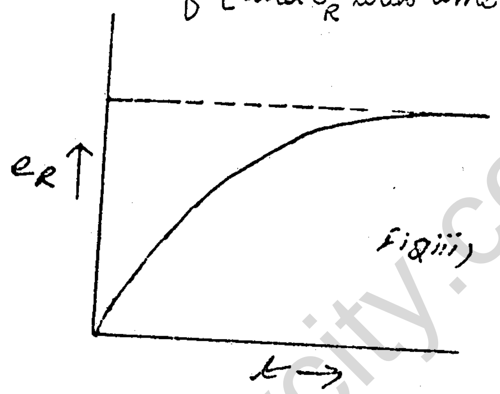
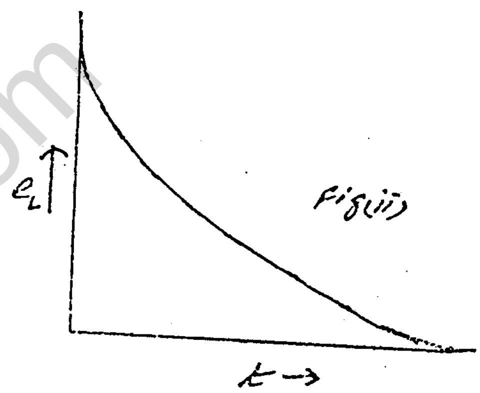


Fig. (ii) and (iii) show the variation of e_L and e_R with time.



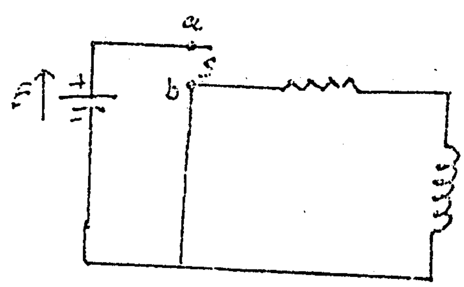


Decay of current in R.L Circuit:

Fig. shows an LR circuit in series with a battery of voltage E. Let us see how current decreases in LR circuit when the battery is disconnected. First close the switch S at (a). The current will start increasing. Wait for a long time so that the current becomes maximum $i_0 = \frac{E}{R}$. Now when the current attains maximum or steady value, open the switch at (a) and close the switch at 'b' immediately. When switch S is closed at 'b' the battery is removed.

$$\therefore e_L + e_R = 0$$

where $e_L = L\frac{di}{dt}$ and $e_R = iR$.

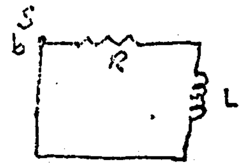


$$\therefore L\frac{di}{dt} + iR = 0.$$

$$L\frac{di}{dt} = -iR$$

$$\frac{di}{dt} = -i\frac{R}{L}.$$

$$\frac{di}{i} = -\frac{R}{L}dt.$$



Integration gives $\ln|i| = -\frac{R}{L}t$

Where A is the constant of integration. To find 'A' we use initial conditions i.e $i = i_0$ at $t = 0$ since initial value of current was i_0 when battery was removed.

$\therefore A = \ln i_0$ and above expression becomes

$$\ln i = -\frac{R}{L}t + \ln i_0$$

$$\ln i - \ln i_0 = -\frac{R}{L}t$$

$$\ln \frac{i}{i_0} = -\frac{R}{L}t$$

$$\frac{i}{i_0} = e^{-\frac{R}{L}t}$$

$$\boxed{i = i_0 e^{-\frac{Rt}{L}}}$$

Putting $\frac{L}{R} = \tau_L$ the time constt

$$\boxed{\therefore\ i = i_0 e^{-t/\tau_L}}$$

This shows that current in LR circuit decreases exponentially.

At $t = \tau_L$, we get

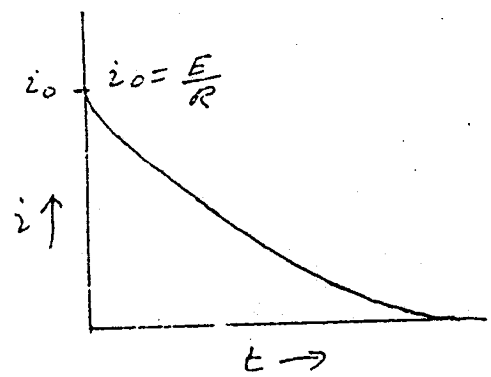
$$i = i_0 e^{-\tau_L/\tau_L} = i_0 e^{-1}$$

$$i = i_0/e = \frac{i_0}{2.718}$$

$$\boxed{i = 0.37 i_0}$$

So inductive time constant is the time after which current decays by 37% of its maximum value. At $t = \infty$, the current $i = 0$. It is clear that decay of current depends on τ_L. The smaller the value of τ_L, the more rapid the decay of current. As $\tau_L = L/R$. So τ_L can be made smaller by increasing the value of R.

Fig shows the decay of current i with time 't'. This curve shows that the current falls to zero after a very long time.



Sample Problem-3

Sample Problem 3 A solenoid has an inductance of 53 mH and a resistance of 0.37 Ω. If it is connected to a battery, how long will it take for the current to reach one-half its final equilibrium value?

Solution. Inductance $L = 53\,mH = 53 \times 10^{-3}\,H$. Resistance $R = 0.37\,\Omega$.

After what time 't' the current decreases to half of max. value.

Let 't' be the time after which $i = i_0/2$.

As $i = E/R\,(1 - e^{-tR/L})$. But $\frac{E}{R} = i_0$

$$\therefore i = i_0\left(1 - e^{-\frac{tR}{L}}\right)$$

Putting $i = i_0/2$.

$$\therefore i_0/2 = i_0\left(1 - e^{-tR/L}\right)$$

$$\frac{1}{2} = 1 - e^{-tR/L}$$

$$e^{-tR/L} = 1 - \frac{1}{2}$$

$$e^{-tR/L} = \frac{1}{2}$$

$$e^{tR/L} = 2$$

Taking natural log on both sides.

$$\ln e^{+tR/L} = \ln 2$$

$$\frac{tR}{L} = \ln 2$$

$$= 0.693$$

$$t = 0.693\,\frac{L}{R}$$

Putting the values of L and R we get

$$t = \frac{0.693 \times 53 \times 10^{-3}}{0.37} = 99.2 \times 10^{-3}$$

$$t = 0.099$$

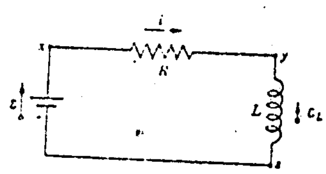
$$\boxed{t = 0.10 \text{ Sec.}}$$ Ans.

Energy Stored in a Magnetic Field
(Energy of Inductor.)

As we know that a current carrying wire is surrounded by a magnetic field. When battery sends a current through an inductor, a variable magnetic field is produced around the inductor. Due to this changing magnetic field a changing magnetic flux passes through the inductor. So a back emf is produced in the inductor. The battery has to do work in sending the current through the inductor. This work done is stored as energy in the magnetic field around the inductor.

Let us find expression for energy stored in the magnetic field surrounding the inductor.

Consider a source of emf E connected to a resistor R and inductor L as shown in the fig.



By using Kirchhoff's 2nd rule we get

$$E = e_R + e_L$$

$$E = iR + L\frac{di}{dt}$$

where $e_R = iR$ and $e_L = L\,\dfrac{di}{dt}$.

Multiplying both sides by 'i' we get

$$\boxed{Ei = i^2R + Li\,\frac{di}{dt}} \quad -(1)$$

Physical Significance:

(i) Suppose charge dq passes through the source of emf in time dt. Then work done by the source is

$$W = E dq. \qquad \because W = Vq.$$

The rate of doing work is $= E\frac{dq}{dt} = Ei$.

So L.H.S of eq.(1) represents the rate at which the work is done by the source of emf. i.e it is the rate at which energy is delivered by the source of emf to the circuit.

(ii) The 2nd term of equation (1) i^2R represents the rate at which energy is dissipated in the resistance R. This energy dissipated appears as internal energy associated with the motion of atoms in the resistor.

(iii) The 3rd term of equation (1) $Li\frac{di}{dt}$ represents the energy stored in the magnetic field of inductor.

If we denote the energy stored in the magnetic field as U_B. Then 3rd term in eq(1) can be written as

$$\frac{d}{dt}U_B = Li\frac{di}{dt}.$$

$$\text{or} \quad \boxed{dU_B = Li\,di} \quad \text{---(2)}$$

The total work done by the battery in storing energy from zero to U_B is obtained by integrating equation (2).

$$\therefore \int_0^{U_B} dU_B = \int_0^i Li\,di.$$

$$U_B = L\int_0^i i\,di$$

$$U_B = L\frac{i^2}{2}.$$

$$\boxed{U_B = \frac{1}{2}Li^2.}$$

This is the expression of energy stored in the magnetic field of inductor having inductance L and carrying current i.

Sample Problem-4

Sample Problem 4 A coil has an inductance of 53 mH and resistance of 0.35 Ω. (*a*) If a 12-V emf is applied, how much energy is stored in the magnetic field after the current has built up to its maximum value? (*b*) In terms of τ_L, how long does it take for the stored energy to reach half of its maximum value?

Solution. Inductance $L = 53\times10^{-3}$ H. Resistance $R = 0.35\,\Omega$

(a) Emf $E = 12$ volt. U_B = Energy stored when current is max = ?

As we know that

$$U_B = \frac{1}{2}Li^2$$

Putting $i = i_0$, the max. current.

$$U_B = \frac{1}{2} L i_0^2 \quad -(1).$$

Now $i_0 = \frac{E}{R} = \frac{12}{0.35} = 34.3$ Amp.

Putting the value of L and i_0 in (1) we get

$$U_B = \frac{1}{2} \times 53 \times 10^{-3} \times 34.3 \times 34.3$$

$$= 31176 \times 10^{-3} = 31.176$$

$$\boxed{U_B = 31 \text{ Joule}}$$

(b). In terms of τ_L after what time the stored energy $U_B = \frac{1}{2}(U_B)_{max}$.

Let 't' be the time after which.

Now $U_B = \frac{1}{2} L i^2$ and $(U_B)_{max} = \frac{1}{2} L i_0^2$.

$$\therefore \frac{1}{2} L i^2 = \frac{1}{2}\left[\frac{1}{2} L i_0^2\right]$$

$$i^2 = \frac{1}{2}\left[i_0^2\right].$$

$$i = \frac{i_0}{\sqrt{2}} \quad -(1).$$

Now $i = \frac{E}{R}(1 - e^{-t/\tau_L})$ and $i_0 = \frac{E}{R}$.

$\therefore$ Equation (1) becomes

$$\frac{E}{R}(1 - e^{-t/\tau_L}) = \frac{E}{R\sqrt{2}}$$

$$1 - e^{-t/\tau_L} = \frac{1}{\sqrt{2}}$$

$$1 - \frac{1}{\sqrt{2}} = e^{-t/\tau_L}.$$

$$e^{-t/\tau_L} = 0.293.$$

$$-t/\tau_L = \ln 0.293$$

$$-\frac{t}{\tau_L} = -1.23$$

$$\frac{t}{\tau_L} = 1.23$$

$$\boxed{t = \tau_L 1.23} \text{ Ans.}$$

So the stored energy reaches half of its maximum value after 1.23 time constants.

Sample Problem. 5

Sample Problem 5 A 3.56-H inductor is placed in series with a 12.8-Ω resistor, an emf of 3.24 V being suddenly applied to the combination. At 0.278 s (which is one inductive time constant) after the contact is made, find (a) the rate P at which energy is being delivered by the battery, (b) the rate P_R at which internal energy appears in the resistor, and (c) the rate P_B at which energy is stored in the magnetic field.

Solution Inductance $L = 3.56$ H $\quad$ $= 12.8\,\Omega$, $E = 3.24$ volt, $t = \tau_L = 0.278$ sec

(a) Rate at which energy is being delivered by the battery = ?

As rate of delivering energy given by

$$P = Ei$$

$$P = 3.24 \times \quad -(1).$$

For i.

$$i = \frac{E}{R}(1 - e^{-t/\tau_L})$$

$$= \frac{3.24}{12.8}(1 - e^{-\tau_L/\tau_L})$$

$$= \frac{3.24}{12.8}(1 - 1/e)$$

$$= 0.253\left(1 - \frac{1}{2.718}\right)$$

$$= 0.253\,(0.632)$$

$$i = 0.16\,A$$

Putting the value of i in (1) we get

$$P = 3.24 \times 0.16$$

$$\boxed{P = 0.5184 \text{ Watt}}$$ Answer.

(b) The rate P_R at which internal energy appears in the resistor.

As P_R is given by

$$P_R = i^2 R.$$

Putting the values of i and R we get.

$$P_R = 0.16 \times 0.16 \times 12.8$$

$$\boxed{P_R = 0.3277 \text{ Watt}}$$ Ans.

(c) The rate P_B at which energy is stored in the magnetic field = ?

As P_B is given by

$$P_B = Li\frac{di}{dt} \quad -(2)$$

As $i = \frac{E}{R}(1 - e^{-t/\tau_L})$

$$\frac{di}{dt} = \frac{E}{R}\left[0 - \left(\frac{-1}{\tau_L}\right)e^{-t/\tau_L}\right].$$

$$= \frac{E}{R}\left(\frac{e^{-t/\tau_L}}{\tau_L}\right)$$

At $t = \tau_L$.

$$= \frac{E}{R\tau_L}e^{-1}$$

$\because \tau_L = \frac{L}{R}$.

$$\frac{di}{dt} = \frac{E}{R\,L/R}e^{-1}$$

$$= \frac{E}{Le} = \frac{3.24}{3.56 \times 2.718}$$

$$\frac{di}{dt} = 0.3348\,A/s$$

Putting the values of L, i & $\frac{di}{dt}$ in (2) we get.

$$P_B = Li\frac{di}{dt}.$$

$$= 3.56 \times 0.16 \times 0.3348$$

$$\boxed{P_B = 0.1907 \text{ Watt}}$$ Ans.

Energy Density and the Magnetic Field

We want to derive an expression for the energy density (energy per unit volume) u_B in a magnetic field.

Consider a very long solenoid having cross-sectional area 'A'. Suppose the core of solenoid is air. Let us consider a portion of length 'l' and volume AL. As the magnetic field due to solenoid lies almost all within the solenoid. So the energy density lies within the volume because the field outside the solenoid is negligible as compared to the field inside it.

As the field within the solenoid is uniform, so energy density within the entire volume must be uniform.

The energy density is written as

$$u_B = \frac{U_B}{AL}.$$

$$\therefore \quad u_B = \frac{\frac{1}{2} L i^2}{AL} \quad \text{---(1)}. \qquad \because U_B = \frac{1}{2} L i^2.$$

Also for a solenoid

$$B = \mu_0 n i.$$

$$\text{or} \quad \boxed{i = \frac{B}{\mu_0 n}}$$

$$\text{and} \quad \boxed{L = \mu_0 n^2 A L}$$

$\therefore$ Expression (1) becomes

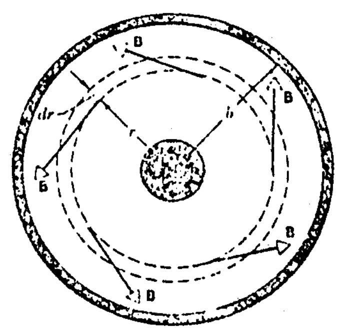
$$u_B = \frac{1}{2} \times \frac{(\mu_0 n^2 A L)}{AL} \left(\frac{B}{\mu_0 n}\right)^2.$$

$$u_B = \frac{B^2}{2\mu_0}$$

This equation gives the energy density at any point (in air, vaccum or in a non magnetic substance) where the magnetic field is uniform. This equation is valid for coil of any shape.

Sample Problem: 6

Sample Problem 6 A long coaxial cable (Fig. 8) consists of two concentric cylindrical conductors with radii a and b, where $b \gg a$. Its central conductor carries a steady current i, and the outer conductor provides the return path. (*a*) Calculate the energy stored in the magnetic field for a length l of such a cable. (*b*) What is the inductance of a length l of the cable?



Solution. In the space b/w two conductors we use Ampere's law and so

$$\oint \vec{B} \cdot \vec{ds} = \mu_0 i.$$

$$\oint B\, ds = \mu_0 i.$$

$$B \oint ds = \mu_0 i$$

$$B(2\pi r) = \mu_0 i$$

$$B = \frac{\mu_0 i}{2\pi r} \quad \text{---(1)}$$

The energy density for points b/w the conductors is given by,

$$u_B = \frac{B^2}{2\mu_0} \quad \text{---(2)}.$$

Putting (1) in (2) we get

$$u_B = \frac{1}{2\mu_0}\left(\frac{\mu_0 i}{2\pi r}\right)^2.$$

$$u_B = \frac{\mu_0 i^2}{8\pi^2 r^2}$$

The energy in a volume dv of shell having radii r and r+dr and length 'l' is given by,

$$dU_B = u_B\, dv.$$

$$= \frac{\mu_0 i^2}{8\pi^2 r^2}(2\pi r l)\, dr$$

$$dU_B = \frac{\mu_0 i^2 l}{4\pi}\left(\frac{dr}{r}\right)$$

The total energy is given by

$$\int dU_B = \frac{\mu_0 i^2 l}{4\pi}\int_a^b \frac{dr}{r}$$

$$U_B = \frac{\mu_0 i^2 l}{4\pi}\left[\ln r\right]_a^b.$$

$$U_B = \frac{\mu_0 i^2 l}{4\pi}(\ln b - \ln a)$$

$$\boxed{U_B = \frac{\mu_0 i^2 l}{4\pi} \ln\frac{b}{a}} \quad \text{Ans.}$$

(b) Inductance $L = ?$

$$\text{As } U_B = \frac{1}{2} L i^2.$$

$$L = \frac{2U_B}{i^2} = \frac{2}{i^2}\left(\frac{\mu_0 i^2 l}{4\pi} \ln\frac{b}{a}\right)$$

$$\boxed{L = \frac{\mu_0 l}{2\pi} \ln\frac{b}{a}} \quad \text{Ans.}$$

Sample Problem: 7

Sample Problem 7 Compare the energy required to set up, in a cube 10 cm on edge, (*a*) a uniform electric field of 1.0×10^5 V/m and (*b*) a uniform magnetic field of 1.0 T. Both these fields would be judged reasonably large but they are readily available in the laboratory.

Solution. Length of each side of circle $= L = 10\text{cm} = .1\text{m}$, $E = 1.0\times10^5$ V/m.

(a) Energy stored in the electric field $= U_E = ?$

$$\text{As } U_E = u_E V_0.$$

$$U_E = \frac{1}{2}\epsilon_0 E^2 V_0$$

$$= \frac{1}{2} 8.85\times10^{-12}\times(1.0\times10^5)^2\times(.1)^3.$$

$$= \frac{8.85}{2} \times 10^{-12}\times10^{10}\times10^{-3} = 4.425\times10^{-5}.$$

$\boxed{U_E = 4.4 \times 10^{-3}\ \text{Joule}}$ Ans

(b) Energy stored in a uniform magnetic field $= U_B = ?$

$B = 1.0T$, $\mu_o = 4\pi \times 10^{-7}\ Wb/A\text{-}m$.

As $U_B = \mu_B V_o$.

$$= \frac{B^2}{2\mu_o} V_o$$

$$= \frac{1 \times 1 \times (10^{-1})^3}{2 \times 4\pi \times 10^{-7}} = \frac{10^4}{25.12}$$

$$= .0398 \times 10^4 = .04 \times 10^4$$

$$= 4 \times 10^2$$

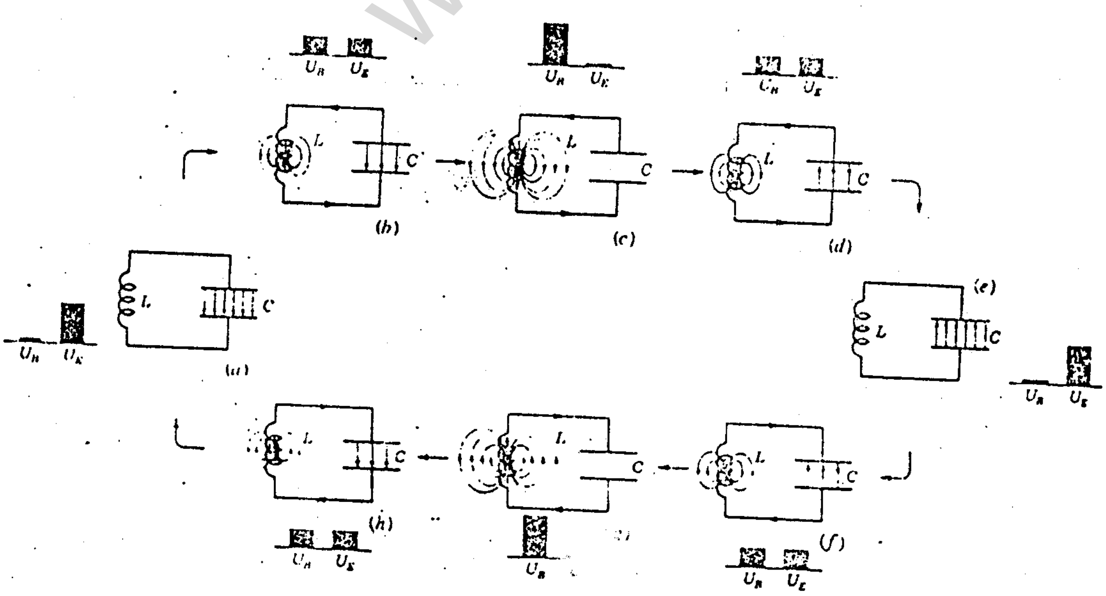
$\boxed{U_B = 400\ J}$ Ans

Electromagnetic Oscillations:
(Qualitative Discussion)

A simple electromagnetic oscillator consists of a capacitor (C) and inductor (L) connected in parallel. This system produces electromagnetic oscillations of frequency depending upon values of L and C.

We shall study the electromagnetic oscillations of a resistanceless oscillator. There are eight stages in one cycle of resistanceless L.C circuit.

(i) Suppose the capacitor is charged by a battery so that it has Charge q_m. Now the battery is removed and an inductor is connected in parallel with the capacitor as shown in fig (a).



There is a potential difference across the plates of C. So the energy is stored in the capacitor in the form of electrostatic energy U_E given by

$$U_E = \frac{1}{2}\frac{q_m^2}{C} \quad -(1)$$

While the magnetic energy $U_B = \frac{1}{2}Li^2$ stored in the inductor is initially zero because the current $i = 0$.

(ii) In fig (b) the capacitor begins to discharge through inductor L +ve charge moves anticlockwise and a current $i = \frac{dq}{dt}$ flows through the inductor. So the energy of capacitor decreases and energy of inductor increases. If no energy is dissipated then decrease in energy of capacitor is exactly equal to increase in energy of inductor s.t total energy remains constt. So electronic field decreases and magnetic field increases. So energy is converted from one form into the other.

(iii) In fig (c) the current becomes max. when capacitor is completely discharged. So electrical energy of capacitor is completely converted into magnetic energy of inductor.

(iv) In fig (d) the energy is flowing from inductor back into capacitor and so magnetic energy starts decreasing and electric field starts increasing

(v) In fig. (v) the capacitor becomes fully charged in the opposite direction as shown in fig (e). So the entire energy of inductor is converted into electrical energy of the capacitor.

(vi) After the capacitor is fully charged, the capacitor now begins to discharge and current starts flowing in the opposite direction as shown in fig (f)

(vii) In fig (g) the capacitor gets fully discharged all electrical energy of capacitor is converted into magnetic energy of inductor.

(viii) In fig (h) the energy flows from inductor back into the capacitor. So the inductor charges the capacitor once again untill the capacitor is fully charged and circuit has gone back to its initial condition as shown in fig (a).

This process goes on indefinetly. This charging and discharging of capacitor takes place again and again and as a result electromagnetic oscillations are produced. The frequency of the oscillations ranges from audio freq (10 HZ) to micro wave frequency (10 GHZ)

Sample Problem: 8

Sample Problem 8 A 1.5-μF capacitor is charged to 57 V. The charging battery is then disconnected and a 12-mH coil is connected across the capacitor, so that LC oscillations occur. What is the maximum current in the coil? Assume that the circuit contains no resistance.

Solution. $C = 1.5\mu F = 1.5\times10^{-6} F$, $V = 57$ Volt. $L = 12$ mH $= 12\times10^{-3}$

Max current $i_m = ?$

By law of conservation of energy

Electrical energy = Magnetic energy.

$$\therefore \frac{1}{2}\frac{q_m^2}{C} = \frac{1}{2}Li_m^2.$$

$$Li_m^2 = q_m^2/C \quad -(1).$$

Now $q_m = CV$

$\therefore$ Expression(1) becomes

$$Li_m^2 = \frac{C^2V^2}{C}$$

$$i_m^2 = \frac{CV^2}{L}$$

$$i_m = V\sqrt{\frac{C}{L}}$$

Putting the values of V, C, L we get.

$$i_m = 57\sqrt{\frac{1.5\times10^{-6}}{12\times10^{-3}}} = 57\sqrt{0.125\times10^{-3}}$$

$$= 57\sqrt{1.25\times10^{-4}} = 57\sqrt{1.25}\times10^{-2}.$$

$$= 63.7\times10^{-2} = 0.637.$$

$$\boxed{i_m = 0.64 \text{ Amp.}}$$ Answer.

Electromagnetic Oscillations:
(Quantitative Analysis)

The total energy of a resistanceless LC circuit at any time is given by using law of conservation of energy as;

$$U = U_B + U_E.$$

$$U = \frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C} \quad -(1)$$

Since the LC circuit is resistanceless so no energy is dissipated. So the total energy of the circuit remains constant i.e $\frac{dU}{dt} = 0$.

$$\therefore \frac{dU}{dt} = \frac{d}{dt}\left[\frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C}\right]$$

$$= 2\cdot\tfrac{1}{2}Li\frac{di}{dt} + 2\frac{1}{2}\frac{q}{C}\frac{dq}{dt}.$$

$$\frac{dU}{dt} = Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt}.$$

$$\therefore \boxed{Li\frac{di}{dt} + \frac{q}{C}\frac{dq}{dt} = 0} \quad -(2) \qquad \because \frac{dU}{dt} = 0.$$

As $\frac{dq}{dt} = i$ and $\frac{d^2q}{dt^2} = \frac{di}{dt}$.

$\therefore$ Expression ( ) becomes.

$$L\frac{dq}{dt}\frac{d^2q}{dt^2} + \frac{q}{C}\frac{dq}{dt} = 0.$$

$$\frac{dq}{dt}\left(L\frac{d^2q}{dt^2} + \frac{q}{C}\right) = 0.$$

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = 0.$$

$$\boxed{\frac{d^2q}{dt^2} + \frac{q}{LC} = 0} \quad -(3)$$

This expression represents the oscillations of LC circuit. This equation is similar to the equation

$$\frac{d^2x}{dt^2} + \frac{K}{m}x = 0.$$

The solution of above equation is

$$x = x_m \cos(\omega t + \varphi)$$

Similarly solution of equation (3) is

$$q = q_m \cos(\omega t + \varphi) \quad -(4).$$

Where ω is the angular frequency of e.m. oscillations.

Now we check whether eq. (4) is a solution of equation (3) or not.

For this purpose we put the values of q and $\frac{d^2q}{dt^2}$ from (4) in (3).

From (3) $\frac{dq}{dt} = -\omega q_m \sin(\omega t + \varphi)$

$\therefore \frac{dq}{dt} = -\omega^2 q_m \cos(\omega t + \varnothing)$

But $q_m \cos(\omega t + \varnothing) = q$ from (3).

$\therefore \frac{d^2q}{dt^2} = -\omega^2 q.$

$\therefore$ Equation (3) becomes

$$-\omega^2 q + \frac{1}{LC} q_m \cos(\omega t + \varnothing) = 0$$

$$-\omega^2 q_m \cos(\omega t + \varnothing) + \frac{1}{LC} q_m \cos(\omega t + \varphi) = 0$$

$$-\omega^2 + \frac{1}{LC} = 0.$$

$$\omega^2 = \frac{1}{LC}.$$

$$\omega = \frac{1}{\sqrt{LC}}$$

So if we find that if we put $\omega = \frac{1}{\sqrt{LC}}$ then eq. (4) is the solution of eq. (3).

As $\omega = \frac{1}{\sqrt{LC}}$

Now time period is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{1/\sqrt{LC}} = 2\pi\sqrt{LC}.$$

$\therefore$ Frequency of oscillation is given by

$$f = 1/T.$$

$$\boxed{f = \frac{1}{2\pi\sqrt{LC}}}$$

This is the expression for frequency of oscillation of a resistanceless LC circuit.

Show that $(U_E)_{max} = (U_B)_{max} = U = U_E + U_B$

As $U = U_E + U_B$ — (A).

Now $U_E = \frac{1}{2}\frac{q^2}{C}$

$= \frac{1}{2C} q_m^2 \cos^2(\omega t + \phi)$ $\quad \because q = q_m \cos(\omega t + \phi)$

$U_E = \frac{q_m^2}{2C} \cos^2(\omega t + \phi)$ — (a)

Now $[\cos^2(\omega t + \phi)]_{max} = 1$.

$\therefore (U_E)_{max} = \frac{q_m^2}{2C}$ — (b).

Now $U_B = \frac{1}{2} L i^2$

$\because q = q_m \cos(\omega t + \phi)$ $\quad \therefore i = \frac{dq}{dt} = -\omega q_m \sin(\omega t + \phi)$

$\therefore U_B = \frac{1}{2} L \frac{1}{LC} q_m^2 \sin^2(\omega t + \phi)$

$U_B = \frac{q_m^2}{2C} \sin^2(\omega t + \phi)$ — (c).

$\because [\sin^2(\omega t + \phi)]_{max} = 1$.

$\therefore (U_B)_{max} = \frac{q_m^2}{2C}$ — (d).

Now putting the values of U_E and U_B from (a) and (c) in (A) we get

$U = \frac{q_m^2}{2C} \cos^2(\omega t + \phi) + \frac{q_m^2}{2C} \sin^2(\omega t + \phi)$.

$= \frac{q_m^2}{2C} [\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)]$.

$= \frac{q_m^2}{2C} (1)$

$U = \frac{q_m^2}{2C}$ — (e).

Hence from (b), (d) and (e) we see that

$(U_E)_{max} = (U_B)_{max} = U$.

So Total energy = Max. energy stored in electric field.

= Max. energy stored in magnetic field.

Fig. shows a plot of U_E and U_B for $\phi = 0$.

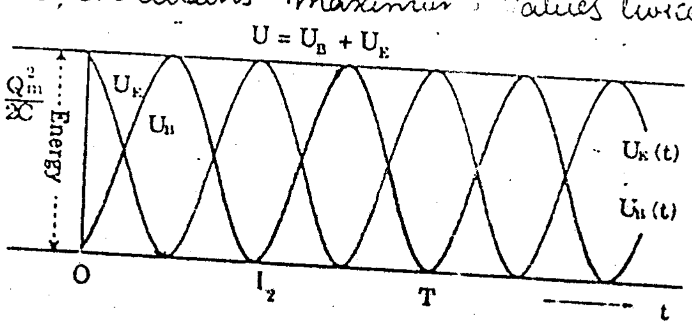
From the figure we see that

(i) $(U_E)_{max} = (U_B)_{max} = \frac{q_m^2}{2C}$.

(ii) The sum of U_E & U_B is constant $= \frac{q_m^2}{2C}$.

(iii) When U_E is maximum, $U_B = 0$ and vice versa.

(iv) U_E and U_B each attains maximum values twice in each cycle.



Fig

21

Sample Problem: 9

(a). *In an oscillating LC circuit what value of charge expressed in terms of the maximum charge is present on the capacitor when the energy is shared equally between the electric and magnetic fields.*

(b) *At what time will this condition occur assuming the capacitor to be fully charged initially? Assume $L = 12\ mH$ and $C = 1.7\ \mu F$.*

Solution. (a) $q = ?$ in term of q_m when $U_E = \frac{1}{2}(U_E)_{max}$.

$$U_E = \frac{q^2}{2C}$$

$$(U_E)_{max} = \frac{q_m^2}{2C}.$$

According to the given condition.

$$U_E = \frac{1}{2}(U_E)_{max}$$

$$\therefore \frac{q^2}{2C} = \frac{1}{2}\left(\frac{q_m^2}{2C}\right)$$

$$q^2 = \frac{1}{2}q_m^2.$$

$$\boxed{q = \frac{q_m}{\sqrt{2}}}$$ Ans.

(b) $L = 12 mH = 12 \times 10^{-3} H$; $C = 1.7 \mu F = 1.7 \times 10^{-6} F$.

At which time t will the condition occur?

Since for $\phi = 0$ at $t = 0$, we have from

$$q = q_m \cos(\omega t + \phi)$$

$$q = q_m$$

At any time 't'

$$q = q_m \cos(\omega t)$$

$$\frac{q_m}{\sqrt{2}} = q_m \cos\omega t. \qquad \because q = \frac{q_m}{\sqrt{2}}$$

$$\cos\omega t = \frac{1}{\sqrt{2}}.$$

$$\omega t = \cos^{-1}\frac{1}{\sqrt{2}} = \pi/4.$$

$$t = \frac{\pi}{4\omega}. \qquad \because \omega = \frac{1}{\sqrt{LC}}$$

$$\therefore t = \frac{\pi}{4\frac{1}{\sqrt{LC}}} = \frac{\pi}{4}\sqrt{LC}.$$

$$t = \frac{3.14}{4}\sqrt{12 \times 10^{-3} \times 1.7 \times 10^{-6}}$$

$$= \sqrt{20.4 \times 10^{-9}} = \sqrt{204 \times 10^{-10}}$$

$$= \sqrt{204} \times 10^{-5} = 11 \times 10^{-5}$$

$$\boxed{t = 1.1 \times 10^{-4}}$$ sec Ans.

Forced Electromagnetic Oscillations and Resonance:

A resistanceless L.C circuit oscillates at the frequency

$$\omega = \frac{1}{\sqrt{LC}}$$

$$\text{or } f = \frac{1}{2\pi\sqrt{LC}}$$ called natural frequency.

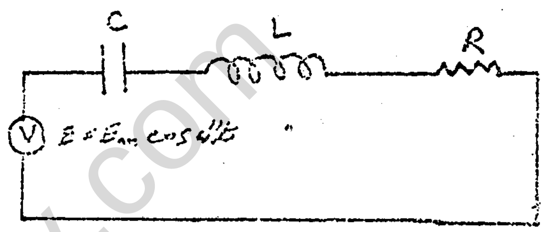
Suppose we drive the circuit with time dependent emf given by

$$E = E_m \cos \omega'' t.$$

Where ω'' is the driving frequency. Such oscillations are called forced oscillations.

Consider a L.C circuit containing a resistance R.

Whatever the natural frequency may be, the forced oscillations take place at the driving frequency ω''.



The current in the circuit is written as

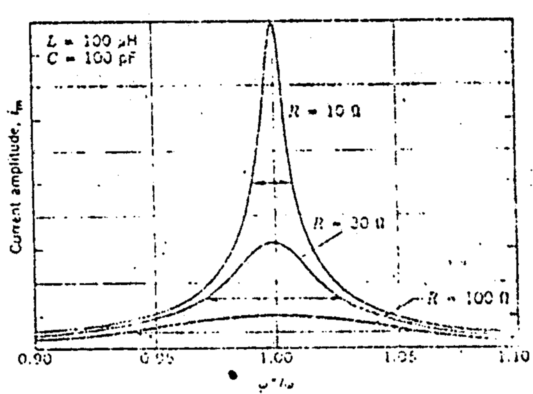
$$i = i_m \sin(\omega'' t - \varphi)$$ where i_m is the maximum value of current. The current in the circuit will be maximum when

$$\omega'' = \omega = \frac{1}{\sqrt{LC}}$$ which is called resonance condition.

Fig. shows three gaps between i_m and $\frac{\omega''}{\omega}$ each for different value of R.

(i) If we see that height of peak is maximum at resonance condition.

(ii) We also see that as R decreases, the resonance peak becomes sharp.



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