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MECHANICS

Mechanics is a branch of Physics which deals with effects of forces on matter in the state of rest or motion.

Vector Operation

Introduction:

(i) Scalars:

The physical quantities which are completely described by their magnitude only are called scalars.

e.g. mass, length, time, density, energy, work, temperature, charge etc.

Scalars can be added, multiplied and subtracted by ordinary rules of algebra.

(ii) Vectors:

The physical quantities which are completely described by their magnitude as well as direction both are called vectors.

e.g. Force, Velocity, acceleration, momentum, torque, electric field, magnetic field etc.

Vectors are added, multiplied and subtracted by vector algebra. However parallel and antiparallel vectors are added ~~or subtr~~ by ordinary algebra.

(iii) Unit Vector:

A vector having unit magnitude and Direction along given vector is called a unit vector.

If we have a vector $\overline{A}$, then a unit vector in the direction of $\vec{A}$ is written as;

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$

Vetor Addition

The process in which two or more than two vectors are added to get a single vector is called vector addition.

Resultant vector

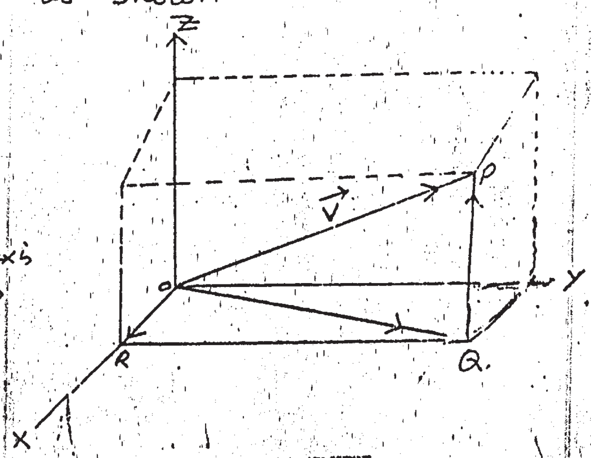
It is sum of two or more vectors which alone has the same effect as the combined effect of all the vectors to be added

Rectangular Components:

"The components of a vector perpendicular to each other are called the rectangular components"

Vector in 3 dimension

Consider a vector $\vec{V}$ represented by $\vec{OP}$ in space as shown. This vector can be decomposed into three mutual perpendicular components along x, y, and z-axis. Let these components are denoted by $\vec{V_x}$, $\vec{V_y}$, and $\vec{V_z}$. These components form three sides of a rectangular parallelopiped as shown



Draw perpendicular PQ on xy-plane, then

$$\vec{OP} = \vec{OQ} + \vec{QP}$$

As vector $\vec{QP}$ is along z-axis

So it is denoted by $\vec{V_z}$

$$\therefore \vec{V} = \vec{OQ} + \vec{V_z} \quad \text{——— (1)}$$

Now draw ⊥ QR on x-axis then

$$\vec{OQ} = \vec{OR} + \vec{RQ}$$

$\vec{OR}$ is along x-axis and is called x-component. It is denoted by $\vec{V_x}$. $\vec{RQ}$ is along y-axis and is called y-component. It is denoted by $\vec{V_y}$.

Then $\vec{OQ} = \vec{V_x} + \vec{V_y}$

∴ equ. 1 becomes

$$\vec{V} = \vec{V_x} + \vec{V_y} + \vec{V_z}$$

If $\hat{i}$, $\hat{j}$ and $\hat{k}$ denote the unit vectors along x-axis, y-axis and z-axis respectively then

$$\vec{V_x} = V_x\hat{i}, \quad \vec{V_y} = V_y\hat{j}, \quad \vec{V_z} = V_z\hat{k}$$

$$\therefore \quad \vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

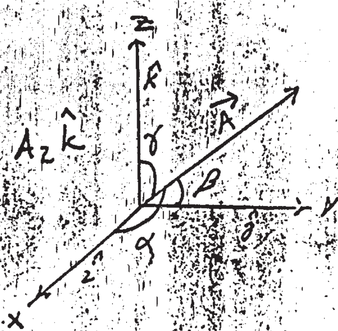
The magnitude of $\vec{V}$ is written as;

$$V = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

x——————x

Direction Cosines

Fig shows a vector $\vec{A}$ in space. Let this vector makes angles α, β and γ with x, y, and z-axis respectively.



Then

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Now unit vector $\hat{A}$ is given by

$$\hat{A} = \frac{\vec{A}}{|A|}$$

$$\therefore \quad \hat{A} = \frac{A_x\hat{i} + A_y\hat{j} + A_z\hat{k}}{|\vec{A}|}$$

OR

$$\hat{A} = \frac{A_x}{|\vec{A}|}\hat{i} + \frac{A_y}{|\vec{A}|}\hat{j} + \frac{A_z}{|\vec{A}|}\hat{k} \quad \text{——— (1)}$$

Now

$$\cos\alpha = \frac{A_x}{|\vec{A}|}$$

$$\cos\beta = \frac{A_y}{|\vec{A}|}$$

$$\cos\gamma = \frac{A_z}{|\vec{A}|}$$

$\therefore$ Equ (1) becomes

$$\hat{A} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

Where $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines of vector $\vec{A}$. These direction cosines fix a vector in space. They are denoted as;

$$\cos\alpha = l$$
$$\cos\beta = m$$
$$\cos\gamma = n$$

So the above expression becomes;

$$\hat{A} = l\hat{i} + m\hat{j} + n\hat{k} \quad \text{——— (2)}$$

This shows that direction cosines are scalar.

Taking magnitude of both sides, we get

$$|\hat{A}| = \sqrt{l^2 + m^2 + n^2}$$

$$1 = \sqrt{l^2 + m^2 + n^2} \qquad \because |\hat{A}| = 1$$

Squaring both sides, we get

$$\boxed{l^2 + m^2 + n^2 = 1}$$

So we find that sum of squares of direction cosines is equal to unity.

x———————x

Problem

Find the length (*i.e.*, the magnitude) of the vector A = 2 i + 3 j + 6 k. Also, calculate the angles which this vector makes with the axes of x, y and z.

Solution.

The magnitude is given by

$$A^2 = A_x^2 + A_y^2 + A_z^2$$

Here

$$A_x = 2 \quad , \quad A_y = 3 \quad A_z = 6$$

$$A^2 = 2^2 + 3^2 + 6^2$$
$$A^2 = 4 + 9 + 36$$
$$A^2 = 49$$
$$\boxed{A = 7} \text{ Ans.}$$

Let $\vec{A}$ makes an angle α, β and γ with x, y, and z-axis respectively. Then

$$\cos\alpha = \frac{A_x}{A} = \frac{2}{7} = 0.285$$
$$\cos\alpha = 0.285$$
$$\alpha = \cos^{-1} 0.285$$
$$\boxed{\alpha = 73^\circ} \text{ Ans.}$$

$$\cos\beta = \frac{A_y}{A} = \frac{3}{7} = 0.428$$
$$\beta = \cos^{-1} 0.428$$

$$\boxed{\beta = 65^\circ} \text{ Ans.}$$

and

$$\cos\gamma = \frac{A_z}{A} = \frac{6}{7} = 0.857$$

$$\gamma = \cos^{-1} 0.857$$

$$\boxed{\gamma = 31^\circ} \text{ Ans.}$$

Problem:

Find the resultant of the vectors :
A = i + 2 j − 8 k
B = 2 i + 2 j + 6 k
C = i + 2 j + 14 k

Solution:

If the resultant vector be $\vec{R}$ then

$$R_x = A_x + B_x + C_x = 1 + 2 + 1 = 4$$

$$R_y = A_y + B_y + C_y = 2 + 2 + 2 = 6$$

$$R_z = A_z + B_z + C_z = -8 + 6 + 14 = 12$$

The magnitude of $\vec{R}$ is given by

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

$$= \sqrt{(4)^2 + (6)^2 + (12)^2}$$

$$= \sqrt{16 + 36 + 144}$$

$$= \sqrt{196}$$

$$\boxed{R = 14} \text{ Ans.}$$

The direction of $\vec{R}$ is given by the angles α, β and γ which $\vec{R}$ makes with x, y, and z-axis respectively.

Now

$$\cos\alpha = \frac{R_x}{R} = \frac{4}{14} = 0.2857$$

$$\alpha = \cos^{-1} 0.2857$$

$$\boxed{\alpha = 73.4^\circ}$$

$$\cos\beta = \frac{R_y}{R} = \frac{6}{14} = 0.4286$$

$$\beta = \cos^{-1} 0.4286$$

$$\boxed{\beta = 64.6^\circ}$$

$$\cos\gamma = \frac{R_z}{R} = \frac{12}{14} = 0.857$$

$$\gamma = \cos^{-1} 0.857$$

$$\boxed{\gamma = 31^\circ}$$

x———x

Problem:

Find the angle between the direction of the vector given by the *difference* of the following two vectors and the z-axis :—

$$A = 5\sqrt{2}\, i + 4\sqrt{2}\, j + 10\, k$$

$$B = 2\sqrt{2}\, i + \sqrt{2}\, j + 2\, k.$$

Solution:

Suppose the difference $\vec{A} - \vec{B} = \vec{R}$

then

$$R_x = (A_x - B_x) = 5\sqrt{2} - 2\sqrt{2} = 3\sqrt{2}$$

$$R_y = (A_y - B_y) = 4\sqrt{2} - \sqrt{2} = 3\sqrt{2}$$

$$R_z = (A_z - B_z) = 10 - 2 = 8$$

$$R^2 = R_x^2 + R_y^2 + R_z^2$$

$$= (3\sqrt{2})^2 + (3\sqrt{2})^2 + (8)^2$$

$$= 18 + 18 + 64$$

$$= 100$$

$$R = 10$$

If $\vec{R}$ makes an angle γ with z-axis then

$$\cos\gamma = \frac{R_z}{R} = \frac{8}{10} = 0.8$$

$$\gamma = \cos^{-1} 0.8$$

$$\boxed{\gamma = 37^\circ} \quad \text{Ans.}$$

x———x

Spherical Polar Coordinates:

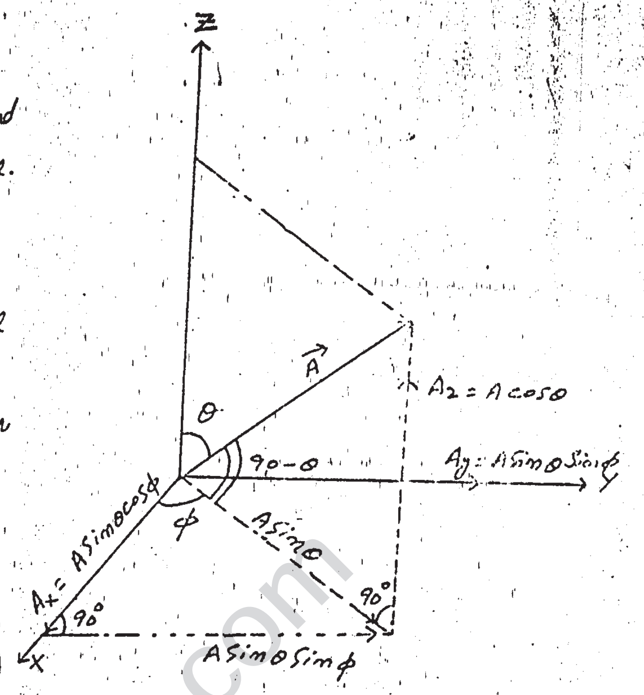
Consider a vector $\vec{A}$ in ~~three direction~~ dimension having components $\vec{A_x}$, $\vec{A_y}$ and $\vec{A_z}$.

Draw projection of $\vec{A}$ on x-y-plane.

The angle θ b/w $\vec{A}$ and z-axis is called polar angle. Then angle ϕ b/w x-axis and projection of $\vec{A}$ in the x-y-plane is called Azimuthal angle.

From the fig. it is clear that

$$A_x = A \sin\theta \cos\phi$$
$$A_y = A \sin\theta \sin\phi$$
$$A_z = A \cos\theta$$



Here A, θ and ϕ or (r, θ, ϕ) are called spherical polar coordinates.

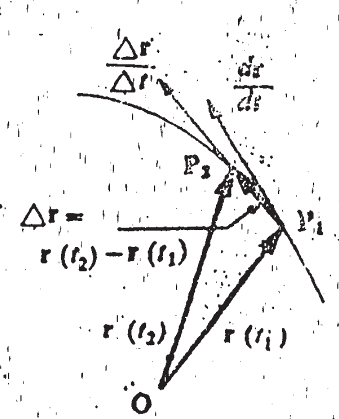
Applications:

The spherical polar coordinates are superior than cartesian coordinates for the study of physical problems. e.g. the gravitational force of the earth on distant objects has the symmetry of a sphere and its properties can be described in an easy way by the help of spherical polar coordinates.

Vector Derivative

Consider a particle moving along a curved path as shown. Its position at any time is given by a vector $\vec{r}(t)$. With the passage of time, the direction & magnitude of this vector may change.

Let $\vec{r}(t_1)$ and $\vec{r}(t_2)$ denote the positions of the particle at times t_1 and t_2 then $\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1)$ is



Fig

a vector is shown. This vector $\Delta\vec{r}$ is represented by the chord $\overrightarrow{P_1P_2}$.

Thus $\Delta\vec{r}$ = chord $\overrightarrow{P_1P_2}$

$\frac{\Delta\vec{r}}{\Delta t}$ is also a vector collinear with $\Delta\vec{r}$. As $\Delta t \to 0$, point P_2 approaches point P_1 and chord P_1P_2 becomes the tangent at P_1. When $\Delta t \to 0$, the ratio $\frac{\Delta\vec{r}}{\Delta t}$ is written as $\frac{d\vec{r}}{dt}$ and is called derivative of $\vec{r}$ w.r.t time. But by definition $\frac{d\vec{r}}{dt}$ is the velocity of the particle.

Thus

$$\frac{d\vec{r}}{dt} = \vec{V}$$

The acceleration being rate of change of velocity is given by

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}$$

In cartesian coordinates

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

Now the position vector of the moving particle can also be written as;

$$\vec{r}(t) = r(t)\hat{r}(t)$$

Then the derivative of $\vec{r}(t)$ is defined as;

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}[r(t)\hat{r}(t)]$$

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{r(t+\Delta t)\hat{r}(t+\Delta t) - r(t)\hat{r}(t)}{\Delta t}$$

By Taylor's Theorem

$$r(t+\Delta t) = r(t) + \frac{dr}{dt}\Delta t$$

and $$\hat{r}(t+\Delta t) = \hat{r}(t) + \frac{d\hat{r}}{dt}\Delta t$$

$$\therefore \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\left[r(t) + \frac{dr}{dt}\Delta t\right]\left[\hat{r}(t) + \frac{d\hat{r}}{dt}\Delta t\right] - r(t)\hat{r}(t)}{\Delta t}$$

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\cancel{r(t)\hat{r}(t)} + r(t)\frac{d\hat{r}}{dt}\Delta t + \frac{dr}{dt}\Delta t\,\hat{r}(t) + \frac{dr}{dt}\frac{d\hat{r}}{dt}(\Delta t)^2 - \cancel{r(t)\hat{r}(t)}}{\Delta t}$$

$$\therefore \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{r(t)\frac{d\hat{r}}{dt}\Delta t + \frac{dr}{dt}\Delta t\,\hat{r}(t) + \frac{dr}{dt}\frac{d\hat{r}}{dt}(\Delta t)^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta t\left[r(t)\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}(t)\right] + (\Delta t)^2\left[\frac{dr}{dt}\frac{d\hat{r}}{dt}\right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{\Delta t}\left[r(t)\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}(t)\right]}{\cancel{\Delta t}} + \frac{(\Delta t)^2\left[\frac{dr}{dt}\frac{d\hat{r}}{dt}\right]}{\cancel{\Delta t}}$$

$$= \lim_{\Delta t \to 0} r(t)\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}t + \Delta t\left(\frac{dr}{dt}\frac{d\hat{r}}{dt}\right)$$

Now taking $\Delta t \to 0$, we get

$$\frac{d\vec{r}}{dt} = r(t)\frac{d\hat{r}}{dt} + \frac{dr}{dt}\hat{r}(t)$$

$$\vec{V} = \frac{dr}{dt}\hat{r}(t) + r(t)\frac{d\hat{r}}{dt} \quad \text{——— (1)} \qquad \because \frac{d\vec{r}}{dt} = \vec{V}$$

This is of the form

$$\frac{d}{dt}(a\vec{b}) = \frac{da}{dt}\vec{b} + a\frac{d\vec{b}}{dt}$$ where a is a scalar & $\vec{b}$ is vector

In eqn (1) $\frac{dr}{dt}$ gives change in magnitude of $\vec{r}$ and

$\frac{d\hat{r}}{dt}$ " " " direction of $\vec{r}$.

Similarly;

(i) $\frac{d}{dt}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$

(ii) $\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$

(iii) $\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt}$

Circular Motion (V. Imp.)

The position vector of a body revolving in a circle is given by;

$$\vec{r}(t) = r\,\hat{r}(t)$$

Suppose that the centre of the circle is at origin O. Now the magnitude of $\vec{r}$ remains constant and the unit vector $\hat{r}(t)$ rotates at a constant rate. A circular motion is an example of a motion in two dimension i.e in a plane. So $\hat{r}(t)$ can be written as;

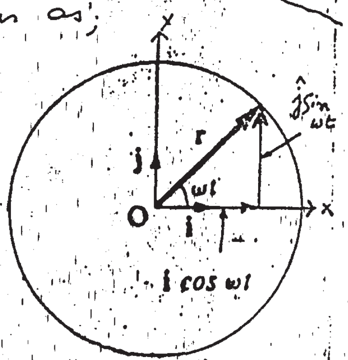
$$\hat{r}(t) = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\because \theta = \omega t$$

$$\therefore \hat{r}(t) = \cos\omega t\,\hat{i} + \sin\omega t\,\hat{j} \quad \text{—— (1)}$$

$\because \vec{r} = x\hat{i} + y\hat{j}$
$\vec{r} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}$
$\vec{r} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$
$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

Where ω is the angular velocity which is constant. The linear velocity of a body is then given as;



$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}(t))$$

$$\vec{V} = \frac{d}{dt}[r\,\hat{r}(t)]$$

$$\vec{V} = r\frac{d}{dt}\hat{r}(t)$$

By putting the value of $\hat{r}(t)$ from equ. (1), we get

$$\vec{V} = r\frac{d}{dt}[\cos\omega t\,\hat{i} + \sin\omega t\,\hat{j}]$$

$$\vec{V} = r(-\sin\omega t\,(\omega)\,\hat{i} + \cos\omega t\,(\omega)\,\hat{j})$$

$\vec{V} = r(-\omega \sin \omega t \hat{i} + \omega \cos \omega t \hat{j})$

$\vec{V} = -r\omega \sin \omega t \hat{i} + r\omega \cos \omega t \hat{j}$

Magnitude of velocity $\vec{V}$ is given by;

$V = \sqrt{r^2\omega^2 \sin^2 \omega t + r^2\omega^2 \cos^2 \omega t}$ $\quad \because \vec{A} = A_x\hat{i} + A_y\hat{j}$

$V = \sqrt{r^2\omega^2(\sin^2 \omega t + \cos^2 \omega t)}$ $\quad A = \sqrt{A_x^2 + A_y^2}$

$V = \sqrt{r^2\omega^2(1)}$

$\boxed{V = r\omega}$ —— (a)

Now for acceleration;

$\vec{a} = \frac{d\vec{v}}{dt}$

$\vec{a} = \frac{d}{dt}(-r\omega \sin \omega t \hat{i} + r\omega \cos \omega t \hat{j})$ $\quad \because r\omega = \text{const.}$

$\therefore \vec{a} = r\omega \frac{d}{dt}(-\sin \omega t(\omega)\hat{i} + \cos \omega t(\omega)\hat{j})$

$= r\omega(-\cos \omega t(\omega)\hat{i} - \sin \omega t(\omega)\hat{j})$

$= r\omega(-\omega \cos \omega t \hat{i} - \omega \sin \omega t \hat{j})$

$= -r\omega\omega(\cos \omega t \hat{i} + \sin \omega t \hat{j})$

$\vec{a} = -r\omega^2 \hat{r}(t)$. $\quad \because \cos \omega t \hat{i} + \sin \omega t \hat{j} = \hat{r}(t)$.

$\vec{a} = -\omega^2 r\hat{r}(t)$

$\therefore \vec{a} = -\omega^2 \vec{r}$ —— (b) $\quad \because r\hat{r}(t) = \vec{r}$

The magnitude of this acceleration is given by;

$a = \omega^2 r$ —— (c)

The -ve sign in equ. (b) shows that acceleration is directed towards the origin i.e towards centre of the circle and equ. (c) shows that magnitude of acceleration $\vec{a}$ is proportional to the distance from the centre 'o' i.e origin.

$$V = r\omega \Rightarrow \omega = \frac{V}{r}$$

$\therefore$ Equ. (ii) becomes
$$a = \frac{V^2}{r^2} r$$

$$\boxed{a = \frac{V^2}{r}}$$ This is called the centripital acceleration

x———x

Scalar Field:

Consider a scalar 'U' eg temperature or density of a medium, which changes from point to point.

"The region in which the scalar varies from point to point is called scalar field."

A scalar function independent of time is called stationary scalar field.

Vector Field:

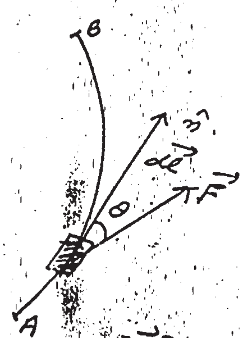
Consider a vector $\vec{V}$ e.g electric intencity which changes from point to point,

"The region in which the vector varies from point to point is called vector field".

x———x

Line Integral:

Consider a vector $\vec{F}$ at point 'p' of curve AB of length 'l' as shown. Take a small element of length 'dl' of curve AB. then the dot product of $\vec{F}$ and $\vec{dl}$ is given by;



$$\vec{F} \cdot \vec{dl} = F \cos\theta \, dl$$
$$= F \cdot \hat{n} \, dl$$

Where $\hat{n}$ is unit vector in the direction of $\vec{dl}$

Integrating the above expression over the entire length 'l', we get

$$\int_A^B \vec{F}\cdot d\vec{l} = \int_l \vec{F}\cdot d\vec{l} = \int_l F\cdot \hat{n}\, dl = \int_l F\, dl \cos\alpha$$

This integral is called line integral of vector along the curve AB.
The common example of line integral is the definition of work.

Surface Integral:

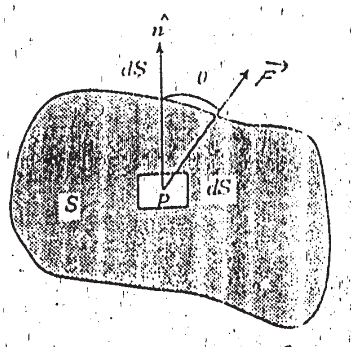
Consider a vector $\vec{F}$ at point 'P' of a surface. Take a small surface of element ds. Then

$$\vec{F}\cdot d\vec{s} = \vec{F}\cdot\hat{n}\, ds = \vec{F}\, ds \cos\theta$$

Where θ is the angle between $\vec{F}$ and outward drawn normal to $d\vec{s}$. $\hat{n}$ is a unit vector along $d\vec{s}$.
By integrating the above expression over whole surface we get;

$$\int\!\!\int_s \vec{F}\cdot d\vec{s} = \int_s \vec{F}\cdot\hat{n}\, ds = \int_s F\, ds \cos\theta .$$

It is called the surface integral of vector $\vec{F}$ over over the whole surface.



If $\vec{F}$ is the velocity of fluid at any point, then $\int \vec{F}\cdot d\vec{s}$ called the rate of flow.
If $\vec{F}$ is the electric or magnatic field strength then $\int \vec{F}\cdot d\vec{s}$ gives the magnetic flux (total flux.)

Volume Integral:

Consider a close surface having uniform volume charge density (ρ). Take a small element of volume dv having mass dm. then

$$\rho = \frac{dm}{dv}$$

$$dm = \rho\, dv$$

Integrating over the whole volume, we get

$$\int dm = \int_V \rho\, dv \qquad \text{where } \int dm = \text{Total mass}$$

$\int dm = \int_V \rho\, dv$ is called volume integral

$$\int dm = \iiint \rho\, dx\, dy\, dz$$

×————×

Operators:

"These are the quantities whose operation function given a new function".

Operators always operate on something placed after them. There are many types of operators,

e.g.
(i) Number operator $(a, 2a, 3a, \ldots, na)$.
(ii) Differential operators $\left(\frac{dx}{dt}, \frac{d^2x}{dt^2} \text{ etc.}\right)$
(iii) Integral Operators $\left(\int dt, \int dx \text{ etc.}\right)$
(iv) Logrithmic Operators $(\log_{10} 100)$
(V) Antilog Operator $(\text{Antilog}(\log_{10} 100))$.

But now we shall introduce a vector differential operator called "del" denoted by $\vec{\nabla}$.

"$\vec{\nabla}$" is written as;

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

It can operate both on scalar and vector.

Operation of $\vec{\nabla}$

We can perform different operations with $\vec{\nabla}$ as shown below,

Gradiant Operation:

"The operation of $\vec{\nabla}$ on a scalar function is called gradient operation. If we have a scalar function U, then

$$\vec{\nabla} U = \vec{grad}\, U$$

Note that $\vec{grad}\, U$ is a vector. In cartesian coordinate $\vec{\nabla} U$ is written as;

$$\vec{\nabla} U = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) U$$

$$\vec{\nabla} U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}$$

(ii) Divergence Operation:

"The operation of $\vec{\nabla}$ on a vector function through dot product is called divergence operation"

If we have a vector function $\vec{V}$, then

$$\vec{\nabla}\cdot\vec{V} = div\,\vec{V}$$

Note that $div\,\vec{V}$ is a scalar. $\vec{\nabla}\cdot\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right)$

$$\vec{\nabla}\cdot\vec{V} = \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

$$\therefore \vec{\nabla}\cdot = div$$

(iii) Curl Operation:

"The operation of $\vec{\nabla}$ on a vector function through cross product is called curl operation."

If we have a vector function $\vec{V}$ then

$$\vec{\nabla}\times\vec{V} = Curl\,\vec{V}$$

Note that $curl\,\vec{V}$ is a vector. In cartesian coordinate, $curl\,\vec{V}$ is written as;

$$\vec{\nabla}\times\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\times\left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right)$$

$$\vec{\nabla}\times\vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$$

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$\therefore Curl = \vec{\nabla}\times$$

Sample Problem

Example : 1.8

A particle moves along a curve whose parametric equations are

$$x = e^{-t},\ y = 2\cos 3t,\ z = 2\sin 3t,$$

where t is the time.
(a) Determine its velocity and acceleration at any time.
(b) Find the magnitude of the velocity and acceleration at $t = 0$.

Solution:

$$x = e^{-t} \quad , \quad y = 2\cos 3t \quad , \quad z = 2\sin 3t$$

(a) $\vec{V} = ?$ $\quad \vec{a} = ?$

As

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}$$

Differentiating both sides w.r.t 't';

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left[e^{-t}\hat{i} + 2\cos 3t\,\hat{j} + 2\sin 3t\,\hat{k}\right]$$

$$\vec{V} = -e^{-t}\hat{i} + 2(-\sin 3t)(3)\hat{j} + 2(\cos 3t)(3)\hat{k}$$

$$\boxed{\vec{V} = -e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}} \text{ — Ans-I}$$

equ.①

Now

$$\vec{a} = \frac{d\vec{V}}{dt}$$

$$\vec{a} = \frac{d}{dt}\left[-e^{-t}\hat{i} - 6\sin 3t\,\hat{j} + 6\cos 3t\,\hat{k}\right]$$

$$= e^{-t}\hat{i} - 6(\cos 3t)(3)\hat{j} + 6(-\sin 3t)(3)\hat{k}$$

$$\boxed{\vec{a} = e^{-t}\hat{i} - 18\cos 3t\,\hat{j} - 18\sin 3t\,\hat{k}} \text{ — Ans-II}$$

equ.②

(b) $V = ?$ at $t = 0$
$a = ?$ at $t = 0$

By putting $t = 0$ in equation ①, we get

$$\vec{V} = -\frac{1}{e^0}\hat{i} - 6\sin 0^\circ\,\hat{j} + 6\cos 0^\circ\,\hat{k}$$

$$\vec{V} = -\hat{i} + 6\hat{k}$$

Magnitude of velocity is given by;

$$V = \sqrt{(-1)^2 + (6)^2}$$

$$V = \sqrt{1+36}$$

$$\boxed{V = \sqrt{37}} \quad \text{Ans.}$$

Now by putting $t = 0$ in equ. (2), we get

$$\vec{a} = \frac{1}{e^0}\hat{i} - 18\cos 0\,\hat{j} - \cancel{18 \sin}\, 0\,\hat{k}$$

$$= \hat{i} - 18\hat{j}$$

$\therefore$ Magnitude of acceleration is given by:

$$a = \sqrt{(1)^2 + (-18)^2}$$

$$a = \sqrt{1 + 324}$$

$$\boxed{a = \sqrt{325}} \quad \text{Ans.}$$

Gradient of a Scalar

Consider a scalar U, e.g. temperature or density of a medium which changes from point to point. The region in which the scalar varies from point to point is called a "scalar field". In such a field U has a definite value at each point. So U is a function of coordinates of point x, y and z.

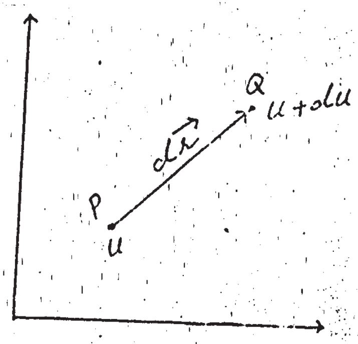
$$\therefore U = U(x, y, z)$$

So if we move from point to point in a scalar field, the value of U changes continuously.

Consider two points 'P' and 'Q' in the scalar field. Let 'U' and $(U + dU)$ be the values of scalar at point P and Q respectively.

Then by calculus

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \quad \text{——(1)}$$



Where dU gives the change in 'U' when we move from 'P' to 'Q'.

Let

$$\overrightarrow{PQ} = d\vec{r}$$

then

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Now equ. (1) can be written as the dot product of two vectors; i.e.

$$dU = \left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right) \quad \text{——(i)}$$

Where $\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)$ is called gradient of scalar and it gives the rate of change of scalar w.r.t. distance along x, y, and z axes. It is written as grad U. It is a vector quantity.

$$\vec{grad}\, U = \frac{\partial U}{\partial x}\hat{i} \; \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}$$

As $\quad d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

So eqn. (2) becomes.

$$dU = \vec{grad}\, U \cdot d\vec{r} \quad \text{——— (3)}$$

The magnitude of grad U is given by;

$$|\vec{grad}\, U| = \sqrt{\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2}$$

For direction of grad U we suppose that P and Q lie in a level surface.

"A level surface is that at which the value of a scalar U remains constant. i.e U has same value at all points of the surface.

$\therefore$ At level surface $\quad du = 0$

$\Rightarrow \quad \vec{grad}\, U \cdot d\vec{r} \quad = 0$

This shows that $\vec{grad}\, U$ is perpendicular to $d\vec{r}$ i.e grad U is perpendicular to the level surface. Grad U can also be written as

$$\vec{grad}\, U = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)U$$

Now $\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial y}\hat{k}\right)$ is a vector differential operator & is denoted by $\vec{\nabla}$ called "del".

$$\therefore \quad \vec{grad}\, U = \vec{\nabla} U$$

Equ. (3) shows that for a given magnitude of $d\vec{r}$ the value of dU becomes maximum, when $d\vec{r}$ is parallel to grad U. Since the scalar product of two vectors is maximum when the angle between the [illegible] is zero.

Hence the maximum value of dU depends on the direction of grad U.

x———x

Problem:

Show that $\vec{F} = -\,\text{grad}\, V$

where V is P.E of the body.

Solution: As we know that

$$dU = \text{grad}\, U \cdot d\vec{r}$$

Put $U = V$;

$$dV = \text{grad}\, V \cdot d\vec{r} \quad \text{——— (1)}$$

As the body moves from high P.E to low P.E, the change in P.E is $= -dV$.

$$-dV = \text{work done on the body}$$

$$-dV = \vec{F} \cdot d\vec{r}$$

OR

$$dV = -\vec{F} \cdot d\vec{r} \quad \text{——— (2)}$$

Comparing eqn (1) and (2) we get,

$$-\vec{F} \cdot d\vec{r} = \text{grad}\, V \cdot d\vec{r}$$

$$\boxed{\vec{F} = -\,\text{grad}\, V}$$

Hence the proof.

x———x

Problem:

The potential energy of a body of mass m placed at a height h above the surface of the earth is mgh. Find the force of gravity (or weight) of the body.

Solution:

Here $PE = V = mgh$ ——— (1)

As $\vec{F} = -\,\text{grad}\, V$

$$\vec{F} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right) \quad \text{——— (a)}$$

Put $h = y$ in above equ. ①, we get

$$V = mgy$$

Putting $V = mgy$ in equ. ⓐ, we get

$$\vec{F} = \frac{-\partial}{\partial x} mgy\,\hat{i} - \frac{\partial}{\partial y} mgy\,\hat{j} - \frac{\partial}{\partial z} mgy\,\hat{k}$$

$$\vec{F} = \frac{-\partial}{\partial y} mgy\,\hat{j}$$

$$\boxed{\vec{F} = -mg\hat{j}}$$

∴ Force of gravity acts in the downward direction.
Mag. of gravity = mg ×——————×

Problem:

Prove that
(i) $\text{grad}\,\phi \cdot \vec{dr} = \vec{\nabla}\phi \cdot \vec{dr}$
(ii) $\text{grad}\,\phi = \vec{\nabla}\phi$

Solution

(i) By definition

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

$$\therefore \vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

Now $$\vec{\nabla}\phi \cdot \vec{dr} = \left(\frac{\partial \varphi}{\partial x}\hat{i} + \frac{\partial \varphi}{\partial y}\hat{j} + \frac{\partial \varphi}{\partial z}\hat{k}\right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$\vec{\nabla}\varphi \cdot \vec{dr} = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz \text{ ——— } ①$$

But we know that

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz$$

But
$$d\varphi = \vec{\text{grad}}\,\varphi \cdot \vec{dr}$$

$$\therefore \quad \vec{\text{grad}}\,\varphi \cdot \vec{dr} = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy\ \frac{\partial \varphi}{\partial z}dz \quad \text{———} \ (2)$$

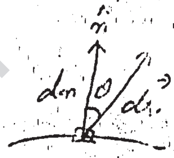
Comparing equ. (1) and equ. (2), we get

$$\boxed{\vec{\text{grad}}\,\varphi \cdot \vec{dr} = \vec{\nabla}\varphi \cdot \vec{dr}}$$ Hence the proof

Alternate Method:

(i) Prove that $\vec{\text{grad}}\,\varphi \cdot \vec{dr} = \vec{\nabla}\varphi \cdot \vec{dr}$

As
$$\vec{\text{grad}}\,\varphi = \frac{\partial \varphi}{\partial n}\hat{n}$$



$$\therefore \ \vec{\text{grad}}\,\varphi \cdot \vec{dr} = \frac{\partial \varphi}{\partial n}\hat{n} \cdot \vec{dr}$$

$$= \frac{\partial \varphi}{\partial n} dr\cos\theta$$

$$\because \cos\theta = \frac{dn}{dr}$$
$$dr\cos\theta = dn$$

$$\therefore \ \vec{\text{grad}}\,\varphi \cdot \vec{dr} = \frac{\partial \varphi}{\partial n}dn \quad \text{———} \ (1)$$

As φ is a function of one variable

$$\therefore \quad \frac{\partial \varphi}{\partial n} = \frac{d\varphi}{dn}$$

$\therefore$ equ. (1) becomes

$$\vec{\text{grad}}\,\varphi \cdot \vec{dr} = \frac{d\varphi}{\cancel{dn}}\cancel{dn}$$

$$\vec{\text{grad}}\,\varphi \cdot \vec{dr} = d\varphi \quad \text{———} \ (2)$$

But
$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz$$

Expression (2) becomes

$$\vec{\text{grad}}\,\varphi \cdot \vec{dr} = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy + \frac{\partial \varphi}{\partial z}dz$$

$$= \left(\frac{\partial \varphi}{\partial x}\hat{i} + \frac{\partial \varphi}{\partial y}\hat{j} + \frac{\partial \varphi}{\partial z}\hat{k}\right)\cdot\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$\boxed{\vec{grad}\,\varphi \cdot \vec{dr} = \vec{\nabla}\varphi \cdot \vec{dr}}$$

Hence proved

(ii) Prove that $\vec{grad}\,\varphi = \vec{\nabla}\varphi$

As it is readily proved that

$$\vec{grad}\,\varphi \cdot \vec{dr} = \vec{\nabla}\varphi \cdot \vec{dr}$$

hence $\boxed{\vec{grad}\,\varphi = \vec{\nabla}\varphi}$ Hence The proof.

Problem:

Show that if a vector is the gradient of a scalar function, then its line integral around a closed path is zero.

Solution:

Let $\vec{F}$ be the vector which is equal to gradient of a scalar function φ.

i.e $\vec{F} = \vec{grad}\,\varphi$

$$\vec{F} \cdot \vec{dr} = \vec{grad}\,\varphi \cdot \vec{dr}$$

Taking line integral on both sides;

$$\int_A^B \vec{F} \cdot \vec{dr} = \int_A^B \vec{grad}\,\varphi \cdot \vec{dr} \quad \text{——— (1)}$$

But $\vec{grad}\,\varphi \cdot \vec{dr} = \left(\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\right) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$\therefore$ $\vec{grad}\,\varphi \cdot \vec{dr} = \left(\frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz\right) = d\varphi$

hence $\vec{grad}\,\varphi \cdot \vec{dr} = d\varphi$

expression (1) becomes $\int_A^B \vec{F} \cdot \vec{dr} = \int_A^B d\varphi$

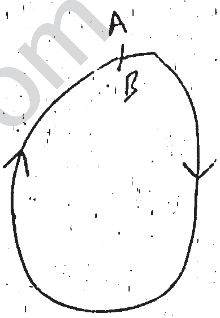
$$\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B d\varphi$$

$$\int_A^B \vec{F}\cdot d\vec{r} = \Big|\varphi\Big|_A^B$$

$$\int_A^B \vec{F}\cdot d\vec{r} = \varphi_B - \varphi_A$$

$\therefore$ Line integral depends upon the values of φ at points A & B. If we have a closed path then the line integral around the closed path is given by;

$$\oint_A^B \vec{F}\cdot d\vec{r} = \oint_A^A d\varphi$$



$$\oint_A^A \vec{F}\cdot d\vec{r} = \Big|\varphi\Big|_A^A$$

$$\oint \vec{F}\cdot d\vec{r} = \varphi_A - \varphi_A$$

$$\boxed{\therefore \oint \vec{F}\cdot d\vec{r} = 0}$$ Hence the result.

x———————x

Problem:

If a scalar function $V = \frac{1}{r}$, where $r = xi + yj + zk$,

show that: $\text{grad } V = \text{grad}\left(\frac{1}{r}\right) = -\frac{r}{r^3}$

<u>Solution:</u>

As

$$\vec{\text{grad}}\, V = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k} \quad \text{———— (a)}$$

Also

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore r^2 = x^2 + y^2 + z^2 \quad \text{———— (b)}$$

$$\because V = \frac{1}{r}$$

$$\therefore \frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{dx}$$

$$\frac{\partial V}{\partial x} = -\frac{1}{r^2}\left(\frac{\partial r}{\partial x}\right) \quad \text{———} \quad (1)$$

Now from equ. (b)

$$r^2 = x^2 + y^2 + z^2$$

$$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x$$

$$r\frac{\partial r}{\partial x} = x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

equ. (1) becomes

$$\frac{\partial V}{\partial x} = \frac{-1}{r^2}\left(\frac{x}{r}\right)$$

$$\boxed{\frac{\partial V}{\partial x} = \frac{-x}{r^3}}$$

Similarly $\boxed{\frac{\partial V}{\partial y} = \frac{-y}{r^3}}$ & $\boxed{\frac{\partial V}{\partial z} = \frac{-z}{r^3}}$

Putting the values of $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ & $\frac{\partial V}{\partial z}$ in equ. (a), we get

$$\text{grad } V = \frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}$$

$$= \frac{-x}{r^3}\hat{i} - \frac{y}{r^3}\hat{j} - \frac{z}{r^3}\hat{k}$$

$$= -\frac{1}{r^3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= -\frac{1}{r^3}\vec{r}$$

$$\therefore \boxed{\text{grad } V = \frac{-\vec{r}}{r^3}} \quad \text{Hence the result.}$$

Problem:

PROBLEM (1.10). Given

$$A = 3x^2y\, i + 4xy\, j + z^2\, k$$

$$B = xz\, i + yz\, j + z^2\, k, \quad \text{find grad}(A \cdot B)$$

Solution:

$$\vec{A}\cdot\vec{B} = (3x^2y\,\hat{i} + 4xy\,\hat{j} + z^2\,\hat{k})\cdot(xz\,\hat{i} + yz\,\hat{j} + z^2\,\hat{k})$$

$$\vec{A}\cdot\vec{B} = 3x^3yz + 4xy^2z + z^4$$

$$\therefore\ \vec{grad}(\vec{A}\cdot\vec{B}) = \vec{\nabla}(\vec{A}\cdot\vec{B})$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(3x^3yz + 4xy^2z + z^4)$$

$$= \hat{i}\frac{\partial}{\partial x}(3x^3yz + 4xy^2z + z^4) + \hat{j}\frac{\partial}{\partial y}(3x^3yz + 4xy^2z + z^4) + \hat{k}\frac{\partial}{\partial z}(3x^3yz + 4xy^2z + z^4)$$

$$= \hat{i}(9x^2yz + 4y^2z) + \hat{j}(3x^3z + 8xyz) + \hat{k}(3x^3y + 4xy^2 + 4z^3)$$

$$\boxed{\vec{grad}(\vec{A}\cdot\vec{B}) = (9x^2yz + 4y^2z)\,\hat{i} + (3x^3z + 8xyz)\,\hat{j} + (3x^3y + 4xy^2 + 4z^3)\,\hat{k}}$$ Ans.

Divergence of a vector: (V.V. Imp.)

Consider a vector $\vec{V}$ e.g velocity of a fluid which changes from point to point. The region in which vector varies from point to point is called vector field. In such a field $\vec{V}$ changes from point to point. So $\vec{V}$ is a function of position coordinates x, y and z. So $\vec{V}$ can be written as;

$$\vec{V} = \vec{V}(x, y, z)$$

If we consider an element of area ΔA in the vector field. Then the scalar product of $\vec{V}$ and ΔA is called the flux of vector field.

$$\therefore\ \text{Flux of vector field} = \vec{V}\cdot\vec{\Delta A}$$

$$= V\Delta A\cos\theta$$

where θ is angle b/w $\vec{V}$ and $\vec{\Delta A}$. If plane of the area is perpendicular to $\vec{V}$, Then $\theta = 0°$

So

$$\text{Flux} = V\Delta A\cos 0°$$

$$\text{Flux} = V\Delta A$$

$$= \frac{\partial V_x}{\partial x}\, dx\, dy\, dz$$

Similarly the net outward flux through ||piped along y-axis and along z-axis gives us;

$$\frac{\partial V_y}{\partial y}\, dx\, dy\, dz \quad \text{and} \quad \frac{\partial V_z}{\partial z}\, dx\, dy\, dz$$

So net outward flux throug ||piped by all the faces is;

$$= \frac{\partial V_x}{\partial x}\, dx\, dy\, dz + \frac{\partial V_y}{\partial y}\, dx\, dy\, dz + \frac{\partial V_z}{\partial z}\, dx\, dy\, dz$$

$$= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dx\, dy\, dz$$

$\because$ $dx\,dy\,dz$ is the volume of ||piped. So the net outward flux per unit volume is;

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

Now by the definition, the net flux per unit volume is called divergence of vector $\vec{V}$. It is denoted by $\operatorname{div} \vec{V}$. Note that it is scalar quantity.

Thus

$$\operatorname{div} \vec{V} = \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

This can also be written as;

$$\operatorname{div} \vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}\ \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right)$$

$$\therefore \operatorname{div} \vec{V} = \vec{\nabla} \cdot \vec{V}$$

If flux entering a volume element is equal to flux leaving the volume element then

$$\operatorname{div} \vec{V} = 0$$

Problem:

Given a position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
Evaluate $\operatorname{div} \vec{r}$.

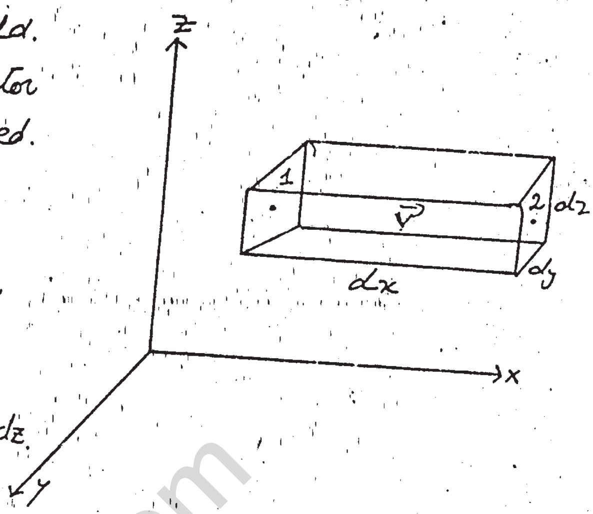
Solution: $\operatorname{div} \vec{r} = \vec{\nabla} \cdot \vec{r}$

$$\operatorname{div} \vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

Physical Significance

Consider a parallelopiped having sides dx, dy, and dz in a vector field. Let $\vec{V}$ be the value of vector at the centre of parallelopiped. Where

$$\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$$



We consider two faces (1) and (2) of the ||piped perpendicular to the x-axis each of area dydz.

The value of x-component of vector $\vec{V}$ at the centre of face-(1) is $= V_x - \frac{\partial}{\partial x} V_x \frac{dx}{2}$

This may be taken as the value for the whole face-(1) because the face is very small.

Hence the flux entering the ||piped through face-(1) is

$$= \left(V_x - \frac{\partial V_x}{\partial x} \frac{dx}{dz} \right) dy\, dz$$

Similarly the value of x-component of vector $\vec{V}$ at the centre of face-(2) is $= V_x + \frac{\partial}{\partial x} V_x + \frac{dx}{2}$

So the flux leaving the face-(2) is

$$= \left(V_x + \frac{\partial}{\partial x} V_x \frac{dx}{2} \right) dy\, dz$$

So the net outward flux through ||piped along x-axis is given;

Flux leaving through face-(2) − Flux entering through face-(1)

$$= \left(V_x + \frac{\partial V_x}{\partial x} \frac{dx}{2} \right) dy\, dz - \left(V_x - \frac{\partial V_x}{\partial x} \frac{dx}{2} \right) dy\, dz$$

$$= \cancel{V_x\, dy\, dz} + \frac{\partial V_x}{\partial x} \frac{dx\, dy\, dz}{2} - \cancel{V_x\, dy\, dz} + \frac{\partial V_x}{\partial x} \frac{dx\, dy\, dz}{2}$$

$$= \cancel{2} \left(\frac{\partial V_x}{\partial x} \frac{dx\, dy\, dz}{\cancel{2}} \right)$$

$$\text{div } \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1$$

$$\boxed{\text{div } \vec{r} = 3} \quad \text{Ans.}$$

Problem:

If $\vec{A} = xz^3\,\hat{i} - 2x^2yz\,\hat{j} + 2yz^4\,\hat{k}$

Find divergence at point $(1, -1, 1)$

Solution

$$\text{div } \vec{A} = \vec{\nabla} \cdot \vec{A}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(xz^3) - \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(2yz^4)$$

$$= z^3 - 2x^2z + 2y(4z^3)$$

$$= z^3 - 2x^2z + 8yz^3$$

$$\text{div } \vec{A} = (1)^3 - 2(1)^2(1) + 8(-1)(1)^3$$

$\because x = 1,\ y = -1,\ z = 1$ according to the given point

$$\boxed{\text{div } \vec{A} = -9} \quad \text{Ans.}$$

Curl of a vector

By definition

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

and $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$

$$\text{Curl } \vec{V} = \vec{\nabla} \times \vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right)$$

$$\text{Curl } \vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$$

$\vec{\nabla} \times \vec{V}$ can also be obtained as:

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$$

Curl $\vec{V}$ is also called rotation of $\vec{V}$ or (Rot $\vec{V}$)

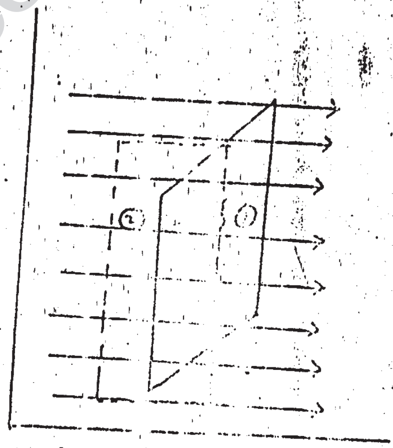
Physical Significance:

It is found that when a vector field is derived as a gradient of a scalar field, then the line integral of such a vector taken over a closed path is zero. i.e

$$\oint \text{grad}\, U \cdot d\vec{l} = 0$$

Where $d\vec{l}$ is the element of path. However there are many vector fields which are not derived as the gradient of scalar field and for which the closed path line integral is not zero.

Fig shows lines of flow of such a vector field e.g linear velocity of a fluid. We consider a rectangular plane area in this field.



When the plane of area is perpendicular to the field as in position ①, the line integral around it is zero.

But when plane of area is parallel to the field as in position ②, then the line integral around the boundry is not zero, but has a definite value. This means the value of a line integral depends upon the orientation of the area.

In general if we place a small vector area of any shape at any point in the vector field and calculate its line integral around the boundry, there will be an orientation of area for which the line integral is max.

"The maximum value of line integral per unit area is called curl of the vector field at that point." Its direction is along the +ve drawn normal.

x ———————— x

Problem:

If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$

Find the curl $\vec{A}$ at the point $(1,-1,1)$.

Solution:

$$\text{Curl } \vec{A} = \vec{\nabla} \times \vec{A}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right]$$
$$+\hat{j}\left[\frac{\partial}{\partial z}(xz^3) - \frac{\partial}{\partial x}(2yz^4)\right]$$
$$+\hat{k}\left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]$$

$$= \hat{i}(2z^4 + 2x^2y) + \hat{j}(3xz^2) + \hat{k}(-4xyz)$$

$$\text{Curl } \vec{A} = (2z^4 + 2x^2y)\,i + (3xz^2)\,j - (4xyz)\,\hat{k}$$

Putting $x = 1$, $y = -1$, $z = 1$, we get

$$\text{Curl } \vec{A} = [2(1)^4 + 2(1)^2(-1)]\hat{i} + [3(1)(1)^2]\hat{j} - [4(1)(-1)(1)]\hat{k}$$

$$\text{Curl } \vec{A} = (2-2)\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\boxed{\text{Curl } \vec{A} = 3\hat{j} + 4\hat{k}}$$ Ans. $\because (2-2)\hat{i} = 0\hat{i} = 0$

Problem:

Given a position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Evaluate $\text{div } \vec{r}$, and $\vec{\nabla} \times \vec{r} = \text{Curl } \vec{r}$

Solution:

As $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\text{div } \vec{r} = \vec{\nabla} \cdot \vec{r}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$\text{div}\,\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1 + 1 + 1$$

$$\therefore \boxed{\text{div}\,\vec{r} = 3}$$

Ans.

Now

$$\text{curl}\,\vec{r} = \vec{\nabla} \times \vec{r}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right] + \hat{j}\left[\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right] + \hat{k}\left[\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right]$$

$$\text{curl}\,\vec{r} = \hat{i}(0-0) + \hat{j}(0-0) + \hat{k}(0-0)$$

$$\therefore \boxed{\text{curl}\,\vec{r} = 0}$$

Ans.

Alternate Curl of a vector

The curl of a vector $\vec{V}$ is given by

$$\text{curl}\,\vec{V} = \vec{\nabla} \times \vec{V}$$

$$\text{curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

It shows that curl of a vector is a vector quantity.

Physical Significance:

Consider a fluid flowing with constant velocity $\vec{V}$ in yz plane along y-axis as shown in fig. The velocity of different layers of fluid goes on increasing.

Hence

$$\text{Curl } \vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\hat{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\hat{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\hat{k}$$

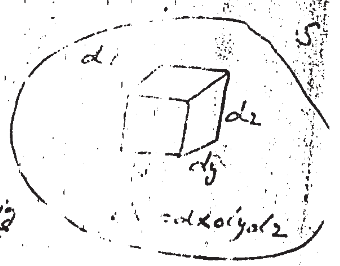
$\therefore$ Rotation of fluid = Curling of fluid = Curl $\vec{V} = \vec{\nabla} \times \vec{V}$

Divergence Theorem
(Gauss's Theorem)

"According to this theorem the surface normal integral of a vector taken over a closed surface is equal to the" volume integral of the divergence of a vector take over the volume enclosed by the surface.

i.e

$$\int_S \vec{V}\cdot d\vec{s} = \int_V \operatorname{div}\vec{V}\, dv$$



$$\int_S \vec{V}\cdot \hat{n}\, ds = \int_V \vec{\nabla}\cdot\vec{V}\, dv$$

Fig

Proof: Consider a small volume element having volume $dv = dx\,dy\,dz$ in a closed surface S. Then by the definition,

$$\operatorname{div}\vec{V} = \vec{\nabla}\cdot\vec{V}$$

$$\operatorname{div}\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right)$$

$$\operatorname{div}\vec{V} = \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

where $\vec{V_x}$, $\vec{V_y}$ & $\vec{V_z}$ are the components of $\vec{V}$. Multiplying both sides by $dx\,dy\,dz$, we get

$$\operatorname{div}\vec{V}\, dx\,dy\,dz = \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dx\,dy\,dz$$

Integrating both the sides; we get

$$\int_V \operatorname{div}\vec{V}\, dv = \iiint\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dx\,dy\,dz$$

$$= \iiint \frac{\partial V_x}{\partial x}\, dx\,dy\,dz + \iiint \frac{\partial V_y}{\partial y}\, dx\,dy\,dz + \iiint \frac{\partial V_z}{\partial z}\, dx\,dy\,dz$$

$$= \int \frac{\partial V_x}{\partial x}\, dx \iint dy\,dz + \int \frac{\partial V_y}{\partial y}\, dy \iint dx\,dz + \int \frac{\partial V_z}{\partial z}\, dz \iint dx\,dy$$

$\because$ V_x, V_y and V_z are function of x, y & z i.e they are function of only one variable. So we can change partial derivative into the total derivative.

$$\therefore \int_V \operatorname{div}\vec{v}\, dv = \int \frac{dV_x}{dx}\, dx \iint dy\,dz + \int \frac{dV_y}{dy}\, dy \iint dx\,dz + \int \frac{dV_z}{dz}\, dz \iint dx\,dy$$

$$= V_x \iint dy\,dz + V_y \iint dx\,dz + V_z \iint dx\,dy$$

$$\therefore \int_V \text{div}\, \vec{V}\, dv = \iint V_x\, dS_x + \iint V_y\, dS_y + \iint V_z\, dS_z$$

$\because dy\,dz = dS_x$ = Area ⊥ to x-axis

$dx\,dz = dS_y$ = " " " y-axis

$dx\,dy = dS_z$ = " " " z-axis

$$= \iint (V_x\, dS_x + V_y\, dS_y + V_z\, dS_z)$$

$$= \iint \vec{V} \cdot d\vec{s}$$

$\because V_x dS_x + V_y dS_y + V_z dS_z = \vec{V} \cdot d\vec{s}$

$$\therefore \int_V \text{div}\, \vec{V}\, dv = \int_S \vec{V} \cdot d\vec{s}$$

or $\boxed{\int_S \vec{V} \cdot d\vec{s} = \int_V \text{div}\, \vec{V}\, dv}$ Hence the proof.

This Theorem enables us to transform a surface integral into a volume integral.

×————×

Problem:

Evaluate $\int_S \vec{r} \cdot \hat{n}\, ds$ where S is a closed surface.

Solution:

By the divergence theorem,

$$\int_S \vec{r} \cdot d\vec{s} = \int_V \vec{\nabla} \cdot \vec{r}\, dv$$

$$= \int_V \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot (x\hat{i} + y\hat{j} + z\hat{k})\, dv$$

$$= \int_V \left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} \right) dv$$

$$= \int_V (1 + 1 + 1)\, dv$$

$$= \int_V 3\, dv$$

$$= 3 \int_V dv$$

$$\boxed{\int_S \vec{r} \cdot \hat{n}\, ds = 3V} \quad \text{Ans.}$$

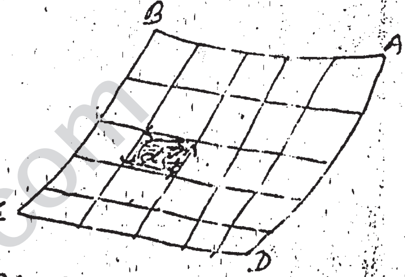
Stoke's Theorem:

According to this theorem "The line integral of a vector function around the closed curve (bounding edge) of a surface is equal to surface normal integral of the curl of vector function over that surface".

If $\vec{V}$ is a vector function then mathematically;

$$\oint \vec{V}.\vec{dl} = \int_S curl\,\vec{V}.\vec{ds}$$

$$\oint \vec{V}.\vec{dl} = \int_S curl\,\vec{V}.\hat{n}\,ds$$



Proof:

Consider a surface enclosed by a curve ABCD. We divide it into large number of small meshes. Let the area of a mesh be $\vec{ds}$.

As curl $\vec{V}$ is the line integral per unit area. So line integral around the boundry of element of area $\vec{ds} = curl\,\vec{V}.\vec{ds}$. Suppose that we take the line integral of all the meshes within the curve ABCD. If we add all these line integrals, the integral along a side of mesh, inside the curve ABCD will be taken twice in opposite directions and so it will cancel away. We are thus left with line integral only along the curve ABCD.

Hence the line integral; $\oint \vec{V}.\vec{dl}$ taken over the curve ABCD and the surface integral $\int_S curl\,\vec{V}.\vec{ds}$ taken over surface ABCD, are equal.

$$\oint \vec{V}.\vec{dl} = \int_S curl\,\vec{V}.\vec{ds}$$

This is called the "Stoke's Theorem". This theorem enables us to transform a line integral into surface integral.

x———————x

Problem:

Prove the following vector identities.

(a) Curl $\vec{grad}\,\varphi = \vec{\nabla}\times\vec{\nabla}\varphi = 0$

(b) div Curl $\vec{A} = \vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0$

Proof

(a) Curl $\vec{grad}\,\varphi = \vec{\nabla}\times\vec{\nabla}\varphi = 0$

Curl $\vec{grad}\,\varphi = \vec{\nabla}\times\vec{\nabla}\varphi$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \varphi \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

Curl $\vec{grad}\,\varphi = \varphi(0) = 0$ $\quad\because$ Two rows of determinent are equal.

$\therefore$ $\boxed{\text{Curl } \vec{grad}\,\varphi = 0}$

Alternative:

As $|\vec{A}\times\vec{A}| = 0$

Curl $\vec{grad}\,\varphi = \vec{\nabla}\times\vec{\nabla}\varphi$

$$= (\vec{\nabla}\times\vec{\nabla})\varphi$$

$$= \vec{0}$$

$\therefore$ $\boxed{\text{Curl } \vec{grad}\,\varphi = 0}$

(b) div Curl $\vec{A} = \vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = 0$

Proof:

As div curl $\vec{A} = \vec{\nabla}\cdot\vec{\nabla}\times\vec{A}$

$$= \vec{\nabla} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \vec{\nabla} \cdot \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{k} \right]$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{k} \right]$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$= \cancel{\frac{\partial^2 A_z}{\partial x \partial y}} - \cancel{\frac{\partial^2 A_y}{\partial x \partial z}} + \cancel{\frac{\partial^2 A_x}{\partial y \partial z}} - \cancel{\frac{\partial^2 A_z}{\partial y \partial x}} + \cancel{\frac{\partial^2 A_y}{\partial x \partial z}} - \cancel{\frac{\partial^2 A_x}{\partial y \partial z}}$$

$$= 0$$

Hence $\boxed{\text{div curl } \vec{A} = 0}$

Alternative

$$\text{div curl } \vec{A} = \vec{\nabla} \cdot \vec{\nabla} \times \vec{A}$$

Because it is just scalar triple product

$$\text{div curl } \vec{A} = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$\boxed{\text{div curl } \vec{A} = 0}$ $\because$ Two rows of determinant are equal

x ————— x

Problem:

Prove that; Curl Curl $\vec{F}$ = grad div $\vec{F} - \nabla^2 \vec{F}$.

Solution

L.H.S
$$\text{curl curl } \vec{F} = \vec{\nabla}\times\vec{\nabla}\times\vec{F}$$
$$= \vec{\nabla}\times(\vec{\nabla}\times\vec{F})$$

$\because \vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$

$\because$ Curl Curl $\vec{F} = \vec{\nabla}\times(\vec{\nabla}\times\vec{F})$
$$= (\vec{\nabla}\cdot\vec{F})\vec{\nabla} - (\vec{\nabla}\cdot\vec{\nabla})\vec{F}$$
$$= (\vec{\nabla}\cdot\vec{F})\vec{\nabla} - \nabla^2\vec{F}$$

Curl Curl $\vec{F} = (\text{div } \vec{F})\vec{\nabla} - \nabla^2\vec{F}$ $\quad \because (\vec{\nabla}\cdot\vec{F}) = \text{div } \vec{F}$

$$\boxed{\text{Curl Curl } \vec{F} = \text{grad div } \vec{F} - \nabla^2 F} \quad \because (\text{div } \vec{F})\vec{\nabla} = \text{grad div } \vec{F}$$

Hence the proof is complete.

Alternative:

Prove by actual expansion.

Curl Curl $\vec{F}$ = grad div $\vec{F} - \nabla^2\vec{F}$

Solution:

L.H.S = Curl curl $\vec{F}$
$$= \vec{\nabla}\times\vec{\nabla}\times\vec{F}$$
$$= \vec{\nabla}\times\left[\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}\right]$$

Its x-component (i.e coefficient of $\hat{i}$) is
$$\frac{\partial}{\partial y}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)$$
$$= \frac{\partial^2 F_y}{\partial x\partial y} - \frac{\partial^2 F_x}{\partial y^2} - \frac{\partial^2 F_x}{\partial z^2} + \frac{\partial^2 F_z}{\partial x\partial z} \quad \text{——— (1)}$$

R.H.S = grad div $\vec{F} - \nabla^2\vec{F}$
$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \nabla^2\vec{F}$$
$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left[\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right] - \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\left[F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right]$$

It x-component (coefficient of $\hat{i}$) is;

$$\frac{\partial}{\partial x}\left[\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}\right]-\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]F_x$$

$$=\cancel{\frac{\partial^2 F_x}{\partial x^2}}+\frac{\partial^2 F_y}{\partial x\partial y}+\frac{\partial^2 F_z}{\partial x\partial z}-\cancel{\frac{\partial^2 F_x}{\partial x^2}}-\frac{\partial^2 F_x}{\partial y^2}-\frac{\partial^2 F_x}{\partial z^2}$$

$$=\frac{\partial^2 F_y}{\partial x\partial y}+\frac{\partial^2 F_z}{\partial x\partial z}-\frac{\partial^2 F_x}{\partial y^2}-\frac{\partial^2 F_x}{\partial z^2}$$

$$R.H.S=\frac{\partial^2 F_y}{\partial y\partial x}-\frac{\partial^2 F_x}{\partial y^2}-\frac{\partial^2 F_x}{\partial z^2}+\frac{\partial^2 F_z}{\partial z\partial x}\quad\text{---------}\quad (2)$$

From equ. (1) and (2) we find that x-component of L.H.S is equal to x-component of R.H.S.

Similarly y-component of L.H.S = y component of R.H.S

& Z- comp. " " = Z comp. " "

Hence $\boxed{\text{Curl curl }\vec{F}=\text{grad div }\vec{F}-\nabla^2\vec{F}}$

Laplace's Operator or Laplacian:

By definition

$$\vec{\nabla}=\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}$$

then $\nabla^2=\vec{\nabla}\cdot\vec{\nabla}$

$$=\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)$$

$$\nabla^2=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$

Where ∇^2 is called Laplace's operator or Laplacian. It is a scalar operator. It can operate both on scalar and vector function.

Laplace's Equation

If operation of ∇^2 on a scalar function is equal to zero then the equation is called Laplace's equ. i.e if we have a scalar function φ then;

$$\boxed{\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0}$$

This is Laplace's equation.

It is linear, homogeneous, 2nd order differencial equation.

x———x

Poison's Equation:

If operation of ∇^2 on a scalar function is equal to some constant with -ve sign, then it is called "Poison's equation."

i.e $$\nabla^2\varphi = \left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right) = -\frac{\rho}{\epsilon_0}$$

Where ρ is density and ϵ_0 is permitivity of free space or

$$\nabla^2\varphi = \frac{-\rho}{\epsilon_0}$$

This is poison's equation.

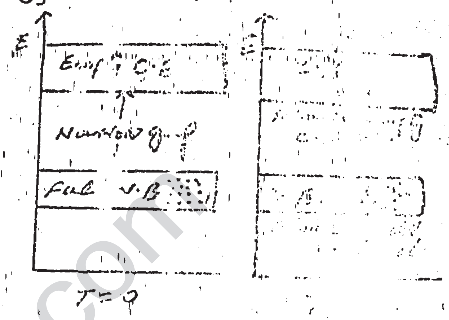
x———x

2 Semi.C.Materials.

Intrinsic S.C: "A pure semiconductor free of impurities is called an intrinsic semiconductor."

They have crystaline structure. The most common examples are (Si) and (Ge).

Figs. shows that the energy bands of pure semi conductors at $T=0$ and $T>0$. We find that the gap between the conduction and valance band is very narrow while for insulators the gap is quite large.



Near absol. zero the cond. band is empty while at a temp. greater than absolute zero, the conduction band is almost empty i.e, it may have some electron.

"A semiconductor material is that whose ... lies between those of conductors and insulators". They are neither good conductors nor good insulators. These are the elements of 4th group of the periodic chart. It means they have a valence of four i.e, have four electrons in their outer most orbit.

The important examples are Ge and Si. These semiconductors are in the form of crystals. Their structure has been studied by the diffraction of x-rays. The Ge and Si crystals are found to be centre-faced cubics as shown. Fig.

If we look at one face of Ge crystals, each atom of Ge is surrounded by its four atoms.

Forbidden E. gap: The separation b/w conduction band & valence band on the energy level diagram is called forbidden E. gap.

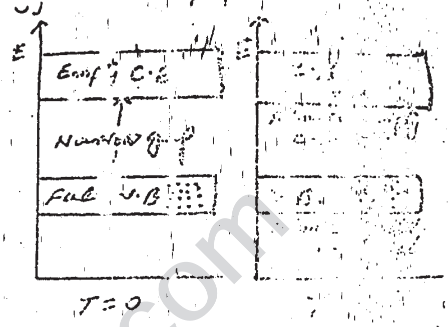
Semiconductor: "A semiconductor material is that ... lies between ... and insulators."

2Semi.C.Materials.

Intrinsic S.C: "A pure semiconductor free of impurities is called an intrinsic semiconductor."

They have crystaline structure. The most common examples are (Si) and (Ge).

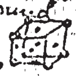
Figs. shows that the energy bands of pure semi conductors at $T=0$ and $T>0$. We find that the gap between the conduction and valance band is very narrow while for insulators the gap is quite large.



Near absol. zero the cond. band is empty while at a temp. greater than absolute zero, the conduction band is almost empty i.e, it may have some electron.

"A semiconductor material is that whose ... lies between those of conductors and insulators". They are neither good conductors nor good insulators. These are the elements of 4th group of the periodic chart. It means they have a valence of four i.e, have four electrons in their outer most orbit.

The important examples are Ge and Si. These semiconductors are in the form of crystals. Their structure has been studied by the diffraction of x-rays. The Ge and Si crystals are found to be centre faced cubics as shown. Fig.



If we look at one face of Ge crystals, each atom of Ge is surrounded by its four atoms.

Forbidden E.gap: The separation b/w conduction band & valence band in energy level diagram is called Forbidden E.gap

Semiconductor: A semic material is that whose ... lies b/w those of conductors and insulators."