Chapter-5

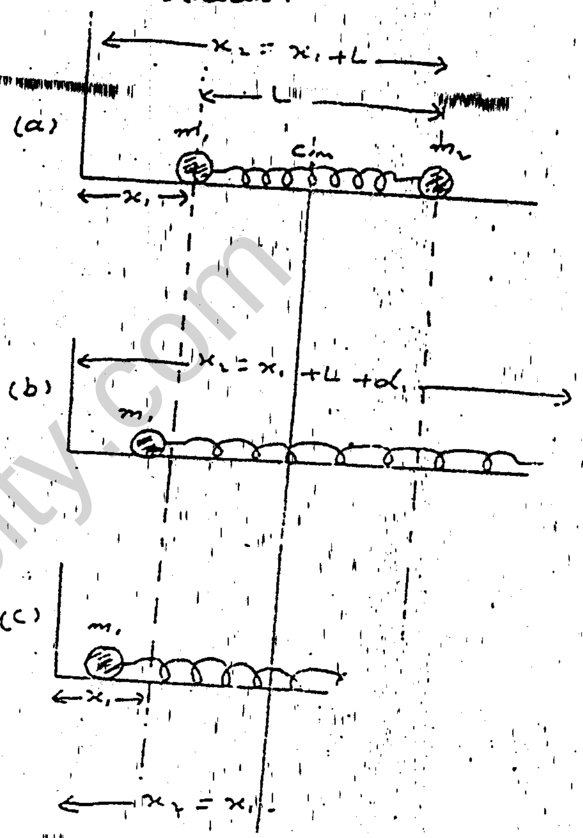
Nasir Pervaiz Bu.
M. Sc. (Physics) M. Phil
Assistant Professor
Govt. College Sargodha.
House No. 338 St. No. 7
Cheema Colony,
Sargodha.

SYSTEM OF PARTICLE

o Particle System:

Consider two bodies are connected by a spring of spring constant K placed on a smooth hori-zontal surface. We suppose that no external force except the spring force is acting on the bodies.

Consider the spring is stretched or compressed from its equilibrium position, it exerts a force on both the bodies. The forces on the two bodies have equal magnitude but opposite in directions. The spring is assumed to be massless. In fig.(b) the spring is given an extension d_1, then energy stored in the spring is given by



$$E_1 = \frac{1}{2} K d_1^2$$

At any instant when ~~energy~~ extension is 'd' the energy is

$$E_2 = U + K$$

$$= \frac{1}{2} K d^2 + \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

By the law of conservation of energy;

$$E_1 = E_2$$

$$\therefore \quad \frac{1}{2} K d_1^2 = \frac{1}{2} K d^2 + \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2$$

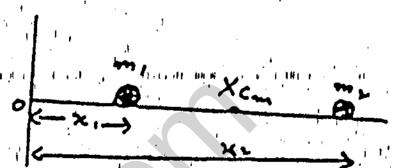
From fig(c),

$$x_2 = x_1 + L + d$$

Where L is length of the spring in equ... ium position. To get a complete solution of this problem consider the centre of the mass of the system. of mass is the point where whole mass of the sy. is concentrated.

For two particle system consisting of masses m_1 & m_2 lying at x_1 and x_2 w.r.t. origin, the position of their centre of mass is given as;

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$



$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{M}$$

where $M = m_1 + m_2$ (Total mass of the system).

The velocity of centre of the mass is given by

$$V_{cm} = \frac{d}{dt} x_{cm}$$

$$= \frac{d}{dt}\left(\frac{m_1 x_1 + m_2 x_2}{M}\right)$$

$$= \frac{1}{M}\frac{d}{dt}(m_1 x_1 + m_2 x_2)$$

$$= \frac{1}{M}\left(m_1 \frac{dx_1}{dt} + m_2 \frac{d}{dt} x_2\right)$$

$$V_{cm} = \frac{1}{M}(m_1 v_1 + m_2 v_2)$$

m. of centre of mass is given by;

$$a_{cm} = \frac{d}{dt} V_{cm}$$

$$= \frac{1}{M}(m_1 V_1 + m_2 V_2)$$

$$= \frac{1}{M}\left(m_1 \frac{d}{dt} V_1 + m_2 \frac{d}{dt} V_2\right)$$

$$a_{cm} = \frac{1}{M}(m_1 a_1 + m_2 a_2) \quad \text{——— (A)}$$

Where a_1 and a_2 are respective accelerations of m_1 & m_2

Now $m_1 a_1 = F_{12}$ (Force exerted by m_2 on m_1)

& $m_2 a_2 = F_{21}$ (" " " m_1 " m_2).

$$\therefore \quad a_{cm} = \frac{1}{M}(F_{12} + F_{21})$$

But by the third Law of motion, we know that

$$F_{12} = -F_{21}$$

$$\therefore \quad a_{cm} = \frac{1}{M}(F_{12} + F_{21})$$

$$= \frac{1}{M}(-\cancel{F_{21}} + \cancel{F_{21}})$$

$$\boxed{a_{cm} = 0}$$

So when no external force acts on the system, the centre of mass ~~a~~ has no acceleration and move with constant velocity which is zero in this case.

General Case: In this case, suppose an external force F_{ext-I} acts on m_1 in addition to F_{12} exerted by m_2 on m_1.

By using the II-law of motion ~~to~~ m_1

$$\vec{F}_{ext-I} + \vec{F}_{12} = m_1 \vec{a}_1 \quad \text{--- (1)}$$

Similarly for m_2

$$\vec{F}_{ext-II} + \vec{F}_{21} = m_2 \vec{a}_2 \quad \text{--- (2)}$$

By adding equ. (1) & equ. (2) we get,

$$\vec{F}_{ext-I} + \vec{F}_{ext-II} + \vec{F}_{12} + \vec{F}_{21} = m_1 \vec{a}_1 + m_2 \vec{a}_2$$

$$\sum \vec{F}_{ext} + \vec{F}_{12} + \vec{F}_{21} = m_1 \vec{a}_1 + m_2 \vec{a}_2$$

Now $\vec{F}_{12} + \vec{F}_{21}$ $\quad \because \vec{F}_{12} = -\vec{F}_{21}$

$$\therefore \sum \vec{F}_{ext} + 0 = m_1 \vec{a}_1 + m_2 \vec{a}_2$$

From equ. A

$$M a_{cm} = m_1 a_1 + m_2 a_2$$

$$\therefore \sum \vec{F}_{ext} = M \vec{a}_{cm}$$

This gives Newton's 2nd Law for a particle of mass 'M' equal to mass of the system.

If $\sum \vec{F}_{ext} = 0$ i.e in the absence of external force, acceleration $a_{cm} = 0$ which means that centre of mass moves with constant velocity, V_{cm}.

2. Many Particles System

Consider a system of N particles of masses $m_1, m_2, m_3, \ldots\ldots m_N$.

The total mass of the system is

$$M = m_1 + m_2 + \ldots\ldots m_N$$

$$M = \sum_{n=1}^{N} m_n$$

The centre of mass of this system is given by

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \ldots\ldots + m_N x_N}{M}$$

$$x_{cm} = \frac{1}{M} \sum_{n=1}^{N} m_n x_n \qquad \text{——— (i)}$$

Similarly

$$y_{cm} = \frac{1}{M} \sum_{n=1}^{N} m_n y_n \qquad \text{——— (ii)}$$

$$z_{cm} = \frac{1}{M} \sum_{n=1}^{N} m_n z_n \qquad \text{——— (iii)}$$

The position vector $\vec{r}(x,y,z)$ of the centre of mass is given as

$$\vec{r}_{cm} = \frac{1}{M}\left(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N\right)$$

$$\vec{r}_{cm} = \frac{1}{M}\sum_{n=1}^{N} m_n\vec{r}_n \quad \text{———} (iv)$$

The velocity of centre of mass is given by:

$$\vec{V}_{cm} = \frac{d}{dt}\vec{r}_{cm}$$

$$= \frac{1}{M}\frac{d}{dt}\left(m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N\right)$$

$$= \frac{1}{M}\left(m_1\frac{d}{dt}\vec{r}_1 + m_2\frac{d}{dt}\vec{r}_2 + \cdots + m_N\frac{d}{dt}\vec{r}_N\right)$$

$$= \frac{1}{M}\left(m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots + m_N\vec{v}_N\right)$$

$$\vec{V}_{cm} = \frac{1}{M}\sum_{n=1}^{N} m_n\vec{v}_n \quad \text{———} (v)$$

The acceleration of the centre of mass is given as:

$$\vec{a}_{cm} = \frac{d}{dt}\vec{V}_{cm}$$

$$= \frac{1}{M}\left(m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + \cdots + m_N\frac{d}{dt}\vec{V}_N\right)$$

$$= \frac{1}{M}\left(m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots + m_N\vec{a}_N\right)$$

$$\vec{a}_{cm} = \frac{1}{M}\sum_{n=1}^{N} m_n\vec{a}_n \quad \text{———} (vi)$$

or

$$M\vec{a}_{cm} = \sum_{n=1}^{N} m_n\vec{a}_n$$

$$M\vec{a}_{cm} = \sum_{n=1}^{N} \vec{F}_n$$

$$M\vec{a}_{cm} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N \quad \text{———} (vii)$$

So the total force acting on a system of particles is equal to the total mass of the system times the acceleration of centre of mass. Equ (vii) represents the

Newton's 2nd Law for a single particle of mass "M" located at the position of centre of mass, moving with a velocity V_{cm} and experiencing an acceleration of a_{cm}.

In general, equ. (vii) can be written as

$$\sum \vec{F}_{ext} = \dot{i}_{cm}$$

In component form,

$$\left.\begin{aligned} \sum \vec{F}_{ext,x} &= M\vec{a}_{cm,x} \\ \sum \vec{F}_{ext,y} &= M\vec{a}_{cm,y} \\ \sum \vec{F}_{ext,z} &= M\vec{a}_{cm,z} \end{aligned}\right\}$$

Hence from the above discussion we arrive t the following impotent result.

i.e overall translational motion of a system of particles can be analysed by using Newton's Laws if all the mass were concentrated at the centre of mass and total external force were applied at that point.

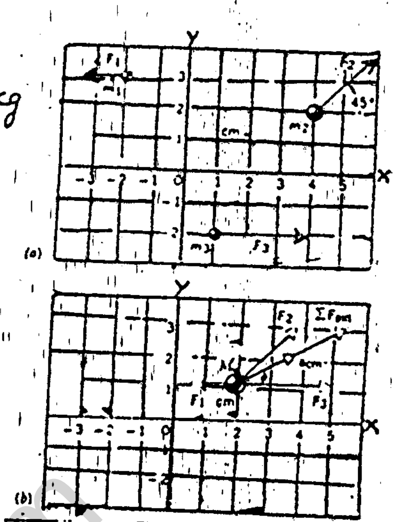
Conversly if

$$\sum \vec{F}_{ext} = 0$$

then the centre of mass of the system move with constant velocity.

Sample Problem: 1

Sample Problem 1 Figure 6a shows a system of three initially resting particles of masses $m_1 = 4.1$ kg, $m_2 = 8.2$ kg, and $m_3 =$ 4.1 kg. The particles are acted on by different external forces, which have magnitudes $F_1 = 6$ N, $F_2 = 12$ N, and $F_3 = 14$ N. The directions of the forces are shown in the figure. Where is the center of mass of this system, and what is the acceleration of the center of mass?



Sol.

$m_1 = 4.1$ kg $m_2 = 8.2$ kg $m_3 = 4.1$ kg

$F_1 = 6$ N $F_2 = 12$ N $F_3 = 14$ N

Centre of mass of the system = ?

Acceleration of centre of mass = ?

(i) Now the total mass of the system

$$M = m_1 + m_2 + m_3$$
$$= 4.1 + 8.2 + 4.1$$
$$M = 16.4 \text{ kg}$$

$$x_{cm} = \frac{1}{M}(m_1 x_1 + m_2 x_2 + m_3 x_3) \quad \text{——— (1)}$$

From fig. $x_1 = -2$ cm

$x_2 = 4$ cm

$x_3 = 1$ cm

Putting the values of $m_1, m_2, m_3, M, x_1, x_2, x_3$ in above equ. (1), we get

$$x_{cm} = \frac{1}{16.4}\left(4.1\times(-2) + 8.2\times 4 + 4.1\times 1\right)$$
$$= \frac{1}{16.4}(-8.2 + 32.8 + 4.1)$$
$$= \frac{1}{16.4}(28.7)$$
$$= 1.75$$

$\boxed{x_{cm} = 1.8 \text{ cm}}$ Ans. I

$$y_{cm} = \frac{1}{M}(m_1 y_1 + m_2 y_2 + m_3 y_3)$$
$$= \frac{1}{16.4}\left(4.1\times 3 + 8.2\times 2 + 4.1\times(-2)\right)$$
$$= \frac{1}{16.4}(12.3 + 16.4 - 8.2)$$
$$= \frac{1}{16.4}(20.5) = 1.25$$

$\because$ from fig.
$y_1 = 3$
$y_2 = 2$
$y_3 = -2$

$\boxed{y_{cm} = 1.3 \text{ cm}}$ Ans. II

For calculating acceleration, we first find the net force F_{ext}.

Now $F_{ext,x} = F_{1x} + F_{2x} + F_{3x}$

$= -F_1 + F_2 \cos\theta + F_3$

$= -6 + 12 \times \cos 45^\circ + 14$

$= -6 + 12(0.707) + 14$

$= -6 + 8.484 + 14$

$= 16.484$

$F_{ext,x} = 16.5\ N$

Similarly $F_{ext,y} = F_{1y} + F_{2y} + F_{3y}$

$= 0 + F_2 \sin\theta + 0$

$= F_2 \sin 45^\circ$

$= 12 \times (0.707)$

$= 8.484$

$F_{ext,y} = 8.5\ N$

The magnitude of F_{ext} is given by

$F_{net} = \sqrt{(F_{ext,x})^2 + (F_{ext,y})^2}$

$= \sqrt{(16.5)^2 + (8.5)^2}$

$= \sqrt{272.25 + 72.25}$

$= \sqrt{344.5}$

$= 18.56$

$\boxed{F_{ext} = 18.6\ N}$

For direction let φ be the angle made by F_{ext} with x-axis

$\tan\varphi = \dfrac{F_{ext,y}}{F_{ext,x}}$

$\varphi = \tan^{-1}\left(\dfrac{F_{ext,y}}{F_{ext,x}}\right)$

$= \tan^{-1}\left(\dfrac{8.5}{16.5}\right)$

$\varphi = \tan^{-1} 0.5151515$

$\varphi = 27.2$

$$\boxed{\varphi = 27^\circ} \quad \text{Ans. III}$$

This is also the direction of acceleration.
Now the magnitude of acceleration is given by;

$$a_{cm} = \frac{F_{ext}}{M}$$

$$= \frac{18.6}{16.4}$$

$$= 1.13$$

$$\boxed{a_{cm} = 1.1 \ m/s^2} \quad \text{Ans IV}$$

x ———————— x

Sample Problem: 2

Sample Problem 2 In the system illustrated in Fig. 7*a*, find the common magnitude of the accelerations of the two blocks. We have already solved this problem, as Sample Problem 8 of Chapter 5, by applying Newton's laws separately to each block. Solve the problem in this case by considering the motion of the center of mass of the two-particle system.

Sol.

(a) Common magnitude of the accelerations of two blocks = ?
We first find the position of centre of mass of the system;

$$x_{cm} = -\frac{m_1}{M}(L - y), \quad y_{cm} = \frac{m_1}{M}(y)$$

Now $V_{cm,x} = \frac{d}{dt} x_{cm}$

$$= \frac{-m_1}{M} \frac{d}{dt}(L - y) \qquad \because L \text{ is total length of chord which is constant.}$$

$$= \frac{+m_1}{M} \frac{dy}{dt}$$

$$V_{cm,x} = \frac{m_1}{M} V$$

Similarly $V_{cm,y} = \frac{d}{dt} y_{cm}$

$$= \frac{m_2}{m} \frac{d}{dt} y$$

$$= \frac{m_2}{m} V$$

$\therefore$ $V_{cm,x} = \frac{m_1}{M} v$ $\qquad V_{cm,y} = \frac{m_2}{M} V$

$$a_{cm,x} = \frac{d}{dt} V_{cm,x}$$

$$= \frac{m_1}{M} \frac{dv}{dt}$$

$$a_{cm,x} = \frac{m_1}{M} a \quad \text{———— (a)}$$

Similarly $a_{cm,y} = \frac{m_2}{M} a$ ———— (b)

where $a = \frac{dv}{dt}$ is common magnitude of the accelerations of two blocks.

Now By Newton's IInd Law along x-axis and along y-axis we get

$$T = M a_{cm,x} \text{ —— (i) along x-axis}$$

$$m_1 g + m_2 g - W - T = M a_{cm,y} \text{ ———— (ii) along y-axis.}$$

Adding (i) and (ii) we get

$$\therefore m_1 g + m_2 g - W = M a_{cm,x} + M a_{cm,y}$$

Putting $W = m_2 g$

$$m_1 g + W - W = M a_{cm,x} + M a_{cm,y}$$

$$m_1 g = M a_{cm,x} + M a_{cm,y}$$

fro $a_{cm,x} = \frac{m_1}{M} a$

from (b) $a_{cm,y} = \frac{m_2}{M} a$

$$m_1 g = M\left(\frac{m_1}{M} a\right) + M\left(\frac{m_2}{M} a\right)$$

$$m_1 g = m_1 a + m_2 a$$

$$m_1 g = a(m_1 + m_2)$$

$$m_1 g = a M \qquad \therefore m_1 + \quad = M$$

$$\frac{m_1}{M} g = a$$

or $\boxed{a = \frac{m_1}{M} g}$

This is the expression for common magnitude of the acceleration of two blocks.

Centre of mass of solid objects:

It is not an easy task to find the centre of mass of a solid object by the relation

$$\vec{r}_{cm} = \frac{1}{M}\sum_{n=1}^{N} m_n \vec{r}_n$$

In case of solids we devide the body into small elements each of mass δm. As these elements become very small then we can write

$$x_{cm} = \frac{1}{M} \lim_{\delta m \to 0} \sum x_n \delta m_n = \frac{1}{M}\int x\, dm \quad \text{---(i)}$$

$$y_{cm} = \frac{1}{M} \lim_{\delta m \to 0} \sum y_n \delta m_n = \frac{1}{M}\int y\, dm \quad \text{---(ii)}$$

$$z_{cm} = \frac{1}{M} \lim_{\delta m \to 0} \sum z_n \delta m_n = \frac{1}{M}\int z\, dm \quad \text{---(iii)}$$

In vector form

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\, dm$$

In terms of uniform volume density

$$\rho = \frac{dm}{dv}$$

$$dm = \rho\, dv$$

So relations (i), (ii) and (iii) become

$$x_{cm} = \frac{1}{M}\int x \rho\, dv = \frac{1}{M}\iiint x \rho\, dx\, dy\, dz$$

$$y_{cm} = \frac{1}{M}\int y \rho\, dv = \frac{1}{M}\iiint y \rho\, dx\, dy\, dz$$

$$z_{cm} = \frac{1}{M}\int z \rho\, dv = \frac{1}{M}\iiint z \rho\, dx\, dy\, dz$$

In vector form

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r} \rho\, dv$$

The process of integration gives an easy method find centre of mass.

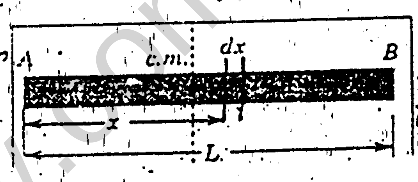
Calculation of centre of mass of Solid Objects

(i) Centre of mass of a uniform rod.

Consider a rod of length L lying along x-axis. Since the rod is uniform, its mass is uniformly distributed along its length.

So the linear mass density of the rod is defined as the mass per unit length. It is given as

Fig



$$\mu = \lim_{\Delta l \to 0} \frac{\Delta m}{\Delta l} = \frac{dm}{dl} = \frac{dm}{dx}$$

or $dm = \mu dl = \mu dx$

The total mass $M = \int dm = \int \mu dl = \int \mu dx$

The position of centre of mass is given by

$$x_{cm} = \frac{1}{M}\int x\, dm$$

$$x_{cm} = \frac{\int x\, dm}{M}$$

$$x_{cm} = \frac{\int_0^L x \mu\, dx}{\int_0^L \mu\, dx} \qquad \because dm = \mu dx$$

$$x_{cm} = \frac{\int_0^L x\, dx}{\int_0^L dx}$$

$$x_{cm} = \frac{\left|\frac{x^2}{2}\right|_0^L}{\left|x\right|_0^L}$$

$$x_{cm} = \frac{L^2/2}{L}$$

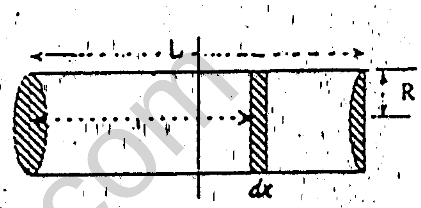
$$x_{cm} = \frac{L}{2}$$

i.e $$x_{cm} = \frac{1}{2} L$$

i.e the centre of mass of a uniform rod lies at its centre.

(ii) Centre of mass of a solid cylinder

Consider a uniform solid cylinder of length 'L' placed along x-axis. As the cylinder is uniform, its mass is uniformly distributed along its length.



Fig

So the volume charge density is defined as mass per unit volume. It is given as;

$$\rho = \underset{\Delta V \to 0}{Lt} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

or $$dm = \rho \, dV$$

The total mass $$M = \int dm = \int \rho \, dV$$

$$M = \int \rho \, dV$$

If 'R' is the radius of the cylinder then volume of element of cylinder dx is given by

$$dV = \pi R^2 dx$$

$$\therefore \quad M = \int \rho \, \pi R^2 dx$$

The position of centre of mass is given by

$$x_{cm} = \frac{1}{M} \int x \, dm$$

$$= \frac{1}{M} \int x \, \rho \, dV$$

$$x_{cm} = \frac{\int x \rho dv}{M}$$

$$x_{cm} = \frac{\int_0^L x \rho \pi r^2 dx}{\int_0^L \rho \pi R^2 dx}$$

$$x_{cm} = \frac{\int_0^L x\, dx}{\int_0^L dx}$$

$$x_{cm} = \frac{\left|x^2/2\right|_0^L}{\left|x\right|_0^L}$$

$$x_{cm} = \frac{L^2/2}{L}$$

$$x_{cm} = \frac{L}{2}$$

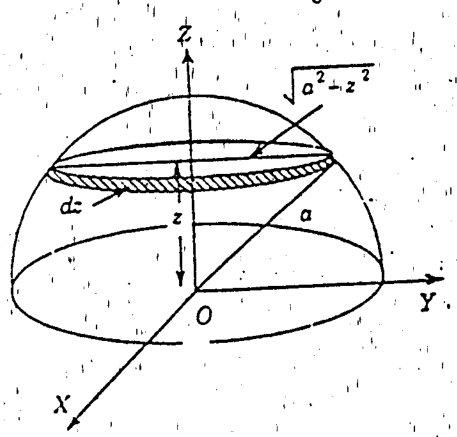
$$x_{cm} = \frac{1}{2} L$$

The centre of mass of uniform solid cylinder lies at the centre of the cylinder.

(iii) Centre of mass of uniform solid Hemisphere

Consider a uniform solid hemisphere having the symmetry axis along z-axis as shown in fig.

The centre of mass lies ~~at~~ on z-axis. Let 'a' be its radius and ρ be the volume density of material of the hemisphere.



The position of centre of mass is given by

$$Z_{cm} = \frac{1}{M} \int Z\, dm$$

$$Z_{cm} = \frac{\int Z\, dm}{\int dm} \qquad \because M = \int dm$$

Now volume density ρ is defined as mass per unit volume,

i.e.

$$\rho = \frac{dm}{dV}$$

$$\therefore \quad dm = \rho\, dv$$

$$\therefore \quad Z_{cm} = \frac{\int z\, dm}{\int dm}$$

$$= \frac{\int z \rho\, dv}{\int \rho\, dv}$$

Let us take the centre of circular base as origin of y and z-axis

We suppose that hemisphere is consist of a number of circular disks of decreasing radii piled one upon the other along z-axis. Let us take one such disc at a height 'z' from the base of hemisphere. Let dz be the thickness of the ~~disk~~, then

Radius of ~~it~~ circular disk, $r = \sqrt{a^2 - z^2}$

Area " " " $ds = \pi r^2 = \pi(a^2 - z^2)$

Volume " " " $dv = \pi r^2 dz$

$= \pi(\ ^2 - z^2)dz$

Mass " " " , $dm = \rho\, dv$

$= \rho\, \pi r^2 dz$

$= \rho\, \pi (a^2 - z^2) dz$

Putting the value of 'dv' in equ. 1 we get

$$Z_{cm} = \frac{\int_0^a z \rho \pi (a^2 - z^2) dz}{\int_0^a \rho \pi (a^2 - z^2) dz}$$

$$= \frac{\int_0^a z (a^2 - z^2) dz}{\int_0^a (a^2 - z^2) dz}$$

$$Z_{cm} = \frac{\int_0^a (a^2 z - z^3)\, dz}{\int_0^a (a^2 - z^2)\, dz}$$

$$= \frac{\left| \frac{a^2 z^2}{2} - \frac{z^4}{4} \right|_0^a}{\left| a^2 z - \frac{z^3}{3} \right|_0^a}$$

$$= \frac{\frac{a^2 a^2}{2} - \frac{a^4}{4}}{a^2 a - \frac{a^3}{3}}$$

$$= \frac{\frac{a^4}{2} - \frac{a^4}{4}}{a^3 - a^3/3}$$

$$= \frac{(2a^4 - a^4)/4}{(3a^3 - a^3)/3}$$

$$= \frac{a^4/4}{2a^3/3}$$

$$= \frac{a^4}{4} \times \frac{3}{2a^3}$$

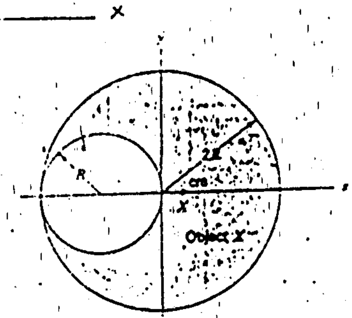
$$\boxed{Z_{cm} = \frac{3a}{8}}$$

This is the position of centre of mass of a uniform solid emisphere.

x —————— —————— x

Sample Problem: 3

Sample Problem 3 Figure 9a shows a circular metal plate of radius $2R$ from which a disk of radius R has been removed. Let us call it object X. Its center of mass is shown as a dot on the x axis. Locate this point.

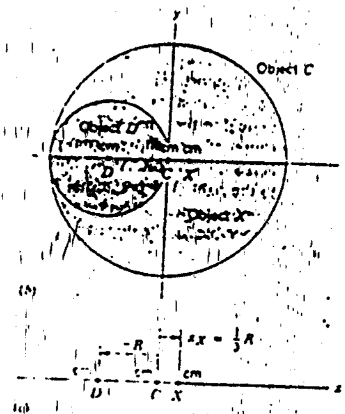


Sol. Fig. 9(a) shows object 'x'.

fig 9(b), the empty space is filled with a disk of radius 'R' and the resultant object is called object 'c'.

The centre of mass of object 'c' is at the origin.

The object 'c' is a combination of object 'x' and object 'D'. The centre of mass of object 'c' is



$$X_c = \frac{m_D x_D + m_x x_x}{m_D + m_x}$$

where 'x_D' is centre of mass of object 'D'

& 'x_x' " " " " object 'x'.

As $x_c = 0$

$$\therefore \quad 0 = m_D x_D + m_x x_x$$

$$m_x x_x = -m_D x_D$$

$$\boxed{x_x = \frac{-m_D x_D}{m_x}} \qquad (1)$$

Now $\quad \frac{m_D}{m_x} = \frac{\text{Area of } D}{\text{Area of } x}$

But $\quad$ Area of x = Area of C − Area of D

$$\therefore \quad \frac{m_D}{m_x} = \frac{\text{Area of } D}{\text{Area of } C - \text{Area of } D}$$

Now $\quad$ Area of 'D' $= \pi R^2$

Area of 'C' $= \pi (2R)^2$

$$\therefore \quad \frac{m_D}{m_x} = \frac{\pi R^2}{\pi (2R)^2 - \pi R^2}$$

$$= \frac{\pi R^2}{4\pi R^2 - \pi R^2}$$

$$= \frac{\pi R^2}{\pi R^2 (4-1)}$$

$$\frac{m_D}{m_x} = \frac{1}{3}$$

Putting the value of $\frac{m_D}{m_x}$ in ① we get

$$x_x = \frac{-m_D\, x_D}{m_x}$$

$$= -\left(\frac{m_D}{m_x}\right) x_D$$

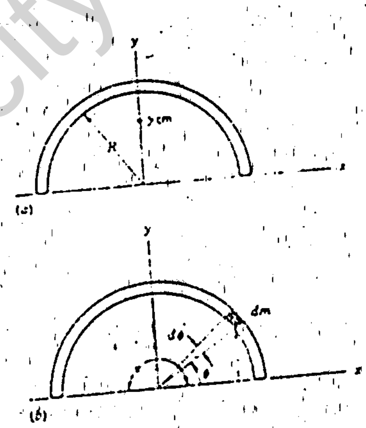
$$= -\frac{1}{3}\, x_D$$

$$x_x = \frac{1}{3} R \qquad \because x_D = -R$$

This is the position of centre of mass of circular plate.

Sample Problem: 4

Sample Problem 4 A thin strip of material is bent into the shape of a semicircle of radius R (Fig. 10). Find its center of mass.



Sol.

Radius of strip $= R$

Centre of mass of strip $= ?$

From fig, it is clear that centre of mass lies on y-axis.

$$\therefore x_{cm} = 0$$

By using the equation

$$y_{cm} = \frac{1}{M}\int y\, dm$$

where $M =$ Total mass of strip

$dm =$ mass of small element of strip

$$\frac{dm}{M} = \frac{d\varphi}{\pi}$$

$$dm = M\frac{d\varphi}{\pi}$$

Now

$$y = R\sin\phi$$

Expression ① becomes

$$y_{cm} = \frac{1}{M}\int (R\sin\varphi)\left(M\frac{d\varphi}{\pi}\right)$$

$$= \int R\sin\varphi\,\frac{d\varphi}{\pi}$$

$$= \frac{R}{\pi}\int \sin\varphi\,d\varphi$$

$$y_{cm} = \frac{R}{\pi}\int_0^{\pi}\sin\varphi\,d\varphi \qquad \text{(As } \pi \text{ changes from } 0 \rightarrow \pi\text{)}$$

$$y_{cm} = \frac{R}{\pi}\Big|-\cos\varphi\Big|_0^{\pi}$$

$$= \frac{R}{\pi}\left(-\cos\varphi + \cos 0\right)$$

$$= \frac{R}{\pi}(1+1)$$

$$= \frac{2R}{\pi}$$

$$y_{cm} = \frac{2R}{3.14}$$

$$y_{cm} = 0.6369\,R$$

$$\boxed{y_{cm} = 0.637\,R} \quad \text{Ans.}$$

x ———————— x

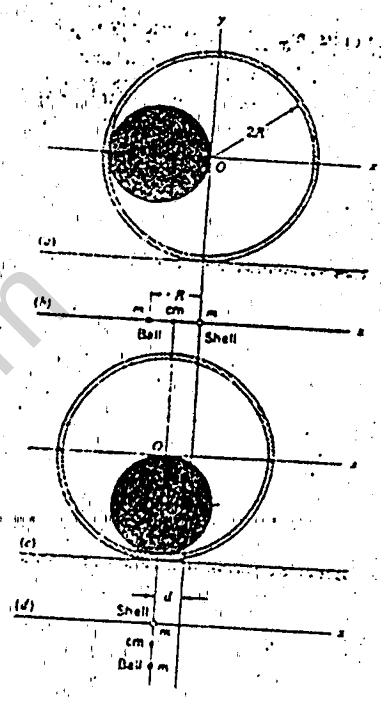
Sample Problem: 5

Sample Problem 5 A ball of mass m and radius R is placed inside a spherical shell of the same mass m and inner radius $2R$. The combination is at rest on a table top as shown in Fig. 11a. The ball is released, rolls back and forth inside, and finally comes to rest at the bottom, as in Fig. 11c. What will be the displacement d of the shell during this process?

Sol.

Fig (a) (on next page) shows that system before the ball is released. Let our origin be at the initial position of centre of the shell.

When the ball released, it rolls back and inside the shell and finally comes to rest. From figure (c) it is clear that shell has moved a distance $d = \frac{1}{2}R$ to the left of origin.



The displacement of the shell is

$$d = \frac{1}{2}R$$

Momentum changes in a system of variable mass

Consider a system having mass M, moving with velocity 'V' at any time 't' in our frame of reference.

After a small interval of time Δt, the system ejects a small amount of mass Δm. So the mass of the system becomes $M - \Delta m$ at time $t + \Delta t$. It should be noted that $\Delta m = -\Delta m$ (decrease in Total mass (M) of the system)

Suppose the system moves with velocity $V + \Delta V$ with remaining mass. Let the velocity of ejected mass Δm be 'U'.

Suppose $\vec{F}_{ext}$ be the external force acting on the system, then

$$\text{Initial momentum } \vec{P_i} = M\vec{V}$$

$$\text{Final momentum } \vec{P_f} = (M - \Delta m)(\vec{V} + \vec{\Delta V}) + (\Delta m)\vec{U}$$

$\therefore$ Change in momentum of the system is given by

$$\Delta\vec{P} = \vec{P_f} - \vec{P_i}$$

$$\Delta\vec{P} = (M - \Delta m)(\vec{V} + \vec{\Delta V}) + (\Delta m)\vec{U} - M\vec{V}$$

As $\Delta m = -\Delta m$, so

$$\Delta\vec{P} = (M + \Delta m)(\vec{V} + \vec{\Delta V}) - \Delta m\vec{U} - M\vec{V}$$

$$= M\vec{V} + M\vec{\Delta V} + \Delta m\vec{V} + \Delta m\vec{\Delta V} - \Delta m\vec{U} - \cancel{M\vec{V}}$$

$$\Delta\vec{P} = M\Delta\vec{V} + \Delta m(\vec{V} - \vec{U}) + \Delta m\Delta\vec{V}$$

Dividing both sides by Δt and taking limit $\Delta t \to 0$ on both sides, we get

$$\lim_{\Delta t \to 0} \frac{\Delta\vec{P}}{\Delta t} = \lim_{\Delta t \to 0} M\frac{\Delta v}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m(\vec{V} - \vec{U})}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m \Delta V}{\Delta t}$$

As $\Delta t \to 0$, $\Delta m \to 0$, $\Delta V \to 0$; hence $\frac{\Delta m}{\Delta t}\Delta V \to 0$

$$\therefore \quad \frac{d\vec{P}}{dt} = M\frac{dv}{dt} + \frac{dm}{dt}(\vec{V} - \vec{U})$$

But $\dfrac{d\vec{P}}{dt} = \vec{F}_{ext}$ by the IInd Law of motion, thus the above equ. becomes

$$\vec{F}_{ext} = M\frac{d\vec{v}}{dt} + \frac{dm}{dt}(\vec{V} - \vec{U}) \quad \text{———} \quad (1)$$

$$\vec{F}_{ext} = M\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt} - \vec{u}\frac{dm}{dt}$$

$$\vec{F}_{ext} = \frac{d}{dt}(M\vec{V}) - \vec{U}\frac{dm}{dt} \quad \text{——— (2)}$$

Equ. (2) can be reduced to single particle form by putting

) $\frac{dm}{dt} = 0$ i.e. m is constant

i) $U = 0$ This is the case with variable mass systems viewed from a special frame of refrence in which ejected mass is at rest.

Equ. (1) can be used to study the rocket motion.

Putting $\vec{U} - \vec{V} = V_{rel}$ = velocity of ejected gases related to the rocket,

Equ. (1) becomes

$$\vec{F}_{ext} = M\frac{d\vec{v}}{dt} + \frac{dm}{dt}(\vec{V} - \vec{U})$$

$$= M\frac{d\vec{v}}{dt} - \frac{dm}{dt}(U - V)$$

$$\vec{F}_{ext} = M\frac{d\vec{v}}{dt} - \frac{dm}{dt}V_{rel}$$

$$M\frac{d\vec{v}}{dt} = \vec{F}_{ext} + \frac{dm}{dt}V_{rel} \quad \text{——— (3)}$$

The last term in equ. (3) gives the rate at which momentum is being transferred into or out of the system. In case of rocket, the term is called thrust.

To make the thrust large, the rocket designers make both the V_{rel} and $\frac{dm}{dt}$ as large as possible.

The rocket equation: The rocket motion takes place under the following two laws.

(i) Third law of motion (ii) Law of conservation of momentum.

The fuel of the rocket is ejected from its tail with very large velocity in the downword direction. As a reaction of the gases the rocket moves up.

In a rocket the rocket and its fuel from a system of two intersecting bodies. As the time parses the mass of the rocket goes on decreasing and its velocity goes on increasing till all its fuel is consumed.

Suppose 'M' is the total mass of rocket at any time 't' moving with velocity '$\vec{V}$'. Suppose $\Delta M = -\Delta m$ be the decrease in mass 'M' of the system. Suppose the rocket moves with velocity $\vec{V}+\Delta\vec{V}$ at time $t+\Delta t$ and the ejected mass moves with velocity $\vec{U}$. then

$$\vec{F}_{ext} = \lim_{\Delta t\to 0}\left[\frac{(M+\Delta M)(\vec{V}+\Delta\vec{V}) + (-\Delta M)\vec{U} - M\vec{V}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{\cancel{M\vec{V}} + M\Delta\vec{V} + \vec{V}\Delta M + \Delta M\Delta\vec{V} - \Delta M\vec{U} - \cancel{M\vec{V}}}{\Delta t}\right]$$

$$= \lim_{\Delta t\to 0}\left(\frac{M\Delta\vec{V} + V\Delta M - \Delta M\vec{U} + \Delta M\Delta V}{\Delta t}\right)$$

When $\Delta t\to 0$, $\Delta M\to 0$ & $\Delta V\to 0$, hence $\Delta M\Delta V\to 0$

$$\therefore \vec{F}_{ext} = \lim_{\Delta t\to 0}\left[\frac{M\Delta\vec{V} + \vec{V}\Delta M - \vec{U}\Delta M}{\Delta t}\right]$$

$$\vec{F}_{ext} = M\frac{d\vec{V}}{dt} + (\vec{V}-\vec{U})\frac{dM}{dt}$$

$$= M\,d\vec{V}/dt - (\vec{U}-\vec{V})\,dM/dt$$

$$\vec{F}_{ext} = M\frac{d\vec{V}}{dt} - \vec{V}_{rel}\frac{dM}{dt}$$

Suppose the rocket is in ~~distinct~~ distant space where external force is acting. Then, $\vec{F}_{ext} = 0$

$$0 = M\frac{d\vec{v}}{dt} - \vec{V}_{rel}\frac{dM}{dt}$$

$$M\frac{d\vec{v}}{dt} = \vec{V}_{rel}\frac{dM}{dt}$$

For the simplicity we suppose that motion of rocket is in one dimension and $\frac{d\vec{v}}{dt}$ is in +ve direction and $\vec{V}_{rel}$ is in −ve direction.

The above expression becomes

$$M\frac{dv}{dt} = -\vec{V}_{rel}\frac{dM}{dt} \quad \text{——— (1)}$$

Because $\frac{dM}{dt}$ is −ve due to decrease in mass. So R.H.S is positive like L.H.S.

Let us calculate the change in velocity of rocket when a certain quantity of fuel m_b burns.

Equ. (1) can be written as

$$d\vec{v} = -V_{rel}\frac{dM}{M} \quad \text{——— (2)}$$

Suppose M_0 = Total original mass of rocket + fuel mass at t=0

& M = Original mass of rocket − mass of fuel burnt during the time 't'.

$$M = M_0 - M_b$$

If v_i & v_f are initial and final velocities of rocket. Then integrating equ. (2) b/w limits of v_i & v_f & M_0 to $M_0 - m_b$, we get

$$\int_{v_i}^{v_f} d\vec{v} = -V_{rel}\int_{M_0}^{M_0 - m_b}\frac{dM}{M}$$

$$|V|_{v_i}^{v_f} = -V_{rel} \left| \log M \right|_{M_0}^{M_0 - m_b}$$

$$V_f - V_i = -V_{rel} \left[\log(M_0 - m_b) - \log M_0 \right]$$

$$V_f - V_i = -V_{rel} \cdot \log\left(\frac{M_0 - m_b}{M_0}\right) \quad \text{——— (3)}$$

This equ. gives change in velocity of the rocket after the fuel of mass m_b burns out.

If we suppose that rocket starts from rest ($V_i = 0$) with mass M_0 and reaches the final velocity V_f when its mass is $M_f = M_0 - m_b$.

Then equ. (3) can be written as

$$V_f = -V_{rel} \log \frac{M_f}{M_0}$$

$$-\frac{V_f}{V_{rel}} = \log \frac{M_f}{M_0}$$

$$e^{-V_f/V_{rel}} = \frac{M_f}{M_0}$$

We can draw similarity b/w rocket fuel system and gun bullet system. In each case the momentum is conserved after the entire system. The term thurst in case of rocket corresponds to the recoil in case of gun.

Sample Problem - 12

Sample Problem 12 A rocket has a mass of 13,600 kg when fueled on the launching pad. It is fired vertically upward and, at burnout, has consumed and ejected 9100 kg of fuel. Gases are exhausted at the rate of 146 kg/s with a speed of 1520 m/s relative to the rocket, both quantities being assumed to be constant while the fuel is burning. (*a*) What is the thrust? (*b*) If we could neglect all external forces, including gravity and air resistance, what would be the speed of the rocket at burnout?

Sol.

Total initial mass $M_0 = 13600$ kg

Mass of ejected fuel $m_b = 9100$ kg

Exhaust velocity, $V_{rel} = 1520\ m/s$

Rate of fuel ejected, $\frac{dM}{dt} = 146\ kg/s$

(a) Thrust = ?

(b) Speed of rocket at burn out $V_f = ?$

As thrust is given by

$$F = V_{rel}\frac{dM}{dt}$$

$$= 1520 \times 146$$

$$= 221920\ N$$

$$= 2.21920 \times 10^5\ N$$

$$\boxed{F = 2.22 \times 10^5\ N}$$

∴ Initial upward force on the rocket = Upward thrust - Initial weight

$$= F - M_0 g$$

$$= 2.2 \times 10^5 - 13600 \times 9.8$$

$$= 2.22 \times 10^5 - 133280$$

$$= 222000 - 133280$$

∴ Initial upward force $= 88720\ N$

Net upward force at burnout = Upward thrust $-(M_0 - m_b)g$

$$= 2.22 \times 10^5 - (13600 - 9100) \times 9.8$$

$$= 222000 - 44100$$

$$= 177900$$

$$= 1.779 \times 10^5$$

∴ Net upward force at burn out $= 1.78 \times 10^5\ N$

(b) As we know that

$$V_f = -V_{rel}\ Log\left(\frac{M_0 - m_b}{M_0}\right)$$

$$V_f = -1520 \log\left(\frac{13600 - 9100}{13600}\right)$$

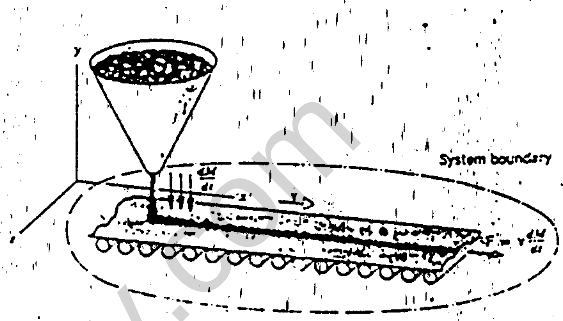
$$= -1520 \log\left(\frac{4500}{13600}\right)$$

$$= -1520 \left[\log 4500 - \log 13600\right]$$

$$= 1520 \left[\log 13600 - \log 4500\right]$$

$$\boxed{V_f = 1681\ m/s}$$ Ans.

Sample Problem 13



Sample Problem 13 Sand drops from a stationary hopper at a rate dM/dt onto a conveyor belt moving with velocity v in the reference frame of the laboratory, as in Fig. 22. What power is required to keep the belt moving at v?

Sol.

Power required to keep belt moving $P_{ext} = ?$

$$F_{ext} = V \frac{dM}{dt}$$

Here $\frac{dM}{dt}$ is +ve because the system is gaining mass with time.

The power supplied by external force is

$$P_{ext} = \vec{F}_{ext} \cdot \vec{V}$$

$$= \left(\vec{V}\frac{dM}{dt}\right) \cdot \vec{V}$$

$$= V^2 \frac{dM}{dt}$$

$\because V$ is constt.

$$\therefore P_{ext} = \frac{d}{dt} MV^2$$

$$= 2 \frac{d}{dt}\left(\frac{1}{2} mV^2\right)$$

$$= 2 \frac{d}{dt} K \qquad \because \frac{1}{2} mV^2 = K$$

$$P_{ext} = 2\frac{dK}{dt} \quad \text{Ans.}$$

So the power required to keep the belt moving is twice the rate of increase of K.E of system.