COLLISIONS

Collision

Collision is an interaction between the bodies that occurs in time Δt that is negligible to the time during which we are observing the system. We can also characterize a collision as an event in which the external forces that may act on the system are negligible compared to impulsive collision forces.

Note:

Collision does not mean, necessarily, physical contact between two bodies.

Elastic Collision

A collision in which the momentum as well as kinetic energies of the colliding bodies is the same before and after the collision.

Kinetic energy is not transformed to the other forms of energy such as internal energy, light, heat ad sound energy etc. also fragmentation of the colliding bodies don't take place.

Inelastic Collision

A collision in which linear momentum of colliding bodies is conserved, but kinetic energy is not conserved.

Some of the kinetic energy may be converted into heat, light and sound etc. also fragmentation of colliding bodies may take place.

Impulsive Forces

A force that acts for a very short interval of time during the collision is called impulsive force. For example when a ball is hit with a bat, impulsive force is applied.

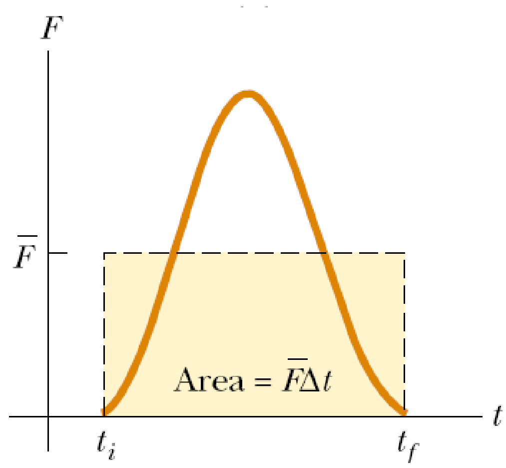
Impulse and Momentum

Consider a collision, in which the impulsive force F(t) is applied on the body. The magnitude of the forces is shown in the figure.

The collision begins at time t_i and ends at time t_f. The force is zero before and after collision.

According to the Newton's 2nd law

$$\vec{F} = \frac{d\vec{P}}{dt}$$



So, the change in momentum $d\vec{P}$ of the particle in time dt during which the force acts on it is

$$d\vec{P} = \vec{F}\,dt$$

Integrating this equation between the initial and final conditions

Momentum P_i at time t_i

Momentum P_f at time t_f

$$\int_{P_i}^{P_f} d\vec{P} = \int_{t_i}^{t_f} \vec{F}\,dt$$

$$\vec{P}_f - \vec{P}_i = \int_{t_i}^{t_f} \vec{F}\,dt \quad \text{-----------} \quad (1)$$

Impulse of a force $\vec{J}$ is defined as product of force and time:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}\,dt \quad \text{----------} \quad (2)$$

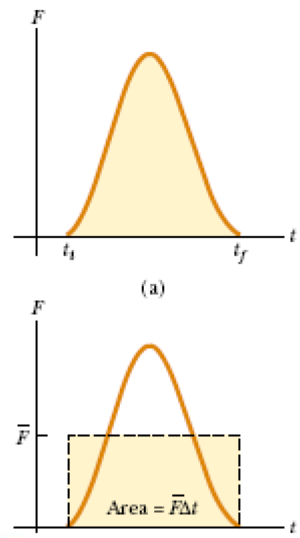
Combining equation (1) and (2), we get:

$$\vec{J} = \vec{P}_f - \vec{P}_i$$

This is Impulse-Momentum Theorem which states that:

The impulse of the net force acting on a particle during a given time interval is equal to the change in momentum of the particle during that interval.

Both the impulse and momentum are vector quantities having same units and dimensions. Impulse-Momentum theorem is very similar to the work energy theorem. The work-energy theorem is a scalar equation dealing change in the kinetic energy of the particle while Impulse-Momentum theorem is a vector equation dealing with the change of momentum of the particles.



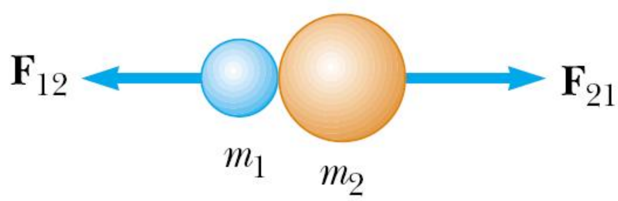
The magnitude of impulse of force is represented by the area under the F(t) curve. The same area is represented by the rectangle of width Δt and height $\vec{F}$. Here $\vec{F}$ is the magnitude of the average force, then

$$\vec{J} = \vec{F}\,\Delta t$$

Conservation of Momentum during Collision

Consider the collision between two particles of masses m_1 and m_2 as shown in the figure.

During brief collision these particles exerts large force on one another.



Let

$\vec{F}_{12}$ is the force exerted on the particle 1 on particle 2.

$\vec{F}_{21}$ is the force exerted on the particle 2 on particle 1.

Let the collision occurs for a time interval $\Delta t = t_f - t_i$. The change in momentum of mass m_1 is

$$\Delta\vec{P}_1 = \int_{t_i}^{t_f} \vec{F}_{12}\,dt = \vec{F}_{12}\,\Delta t \quad \text{----------} \quad (1)$$

Similarly, the change in momentum of mass m_2 is

$$\Delta\vec{P}_2 = \int_{t_i}^{t_f} \vec{F}_{21}\,dt = \vec{F}_{21}\,\Delta t \quad \text{----------} \quad (2)$$

As $\vec{F}_{12}$ and $\vec{F}_{21}$ are equal but opposite, then

$$\vec{F}_{12} = -\vec{F}_{21}$$

Then from equation (1),

$$\Delta\vec{P}_1 = \vec{F}_{12}\,\Delta t = -\vec{F}_{21}\,\Delta t = -\Delta\vec{P}_2$$

$$\Rightarrow \Delta\vec{P}_1 + \Delta\vec{P}_2 = 0$$

So, the total change on momentum $\Delta\vec{P}_1 + \Delta\vec{P}_2 = \Delta\vec{P}$ is zero. Hence we conclude that

If there is mo external forces, the total momentum of the two particle system is not changed by collision.

The is the law of conservation of linear momentum, so

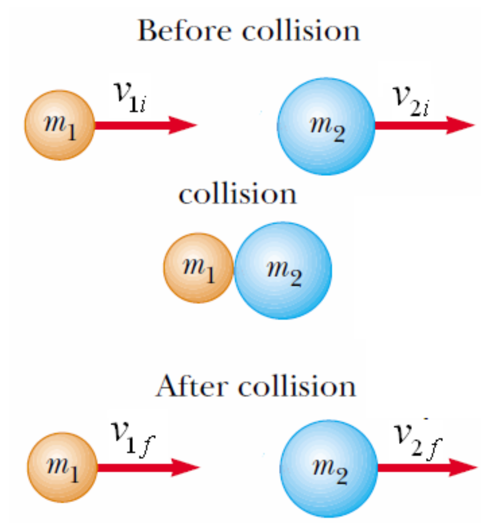
Momentum of the system before collision = Momentum of the system after collision

Elastic Collision in One Dimension

Consider two spherical objects of an isolated system having masses m_1 and m_2, moving with initial velocities v_{1i} and v_{2i}, respectively, along the same direction (say x-axis). After collision, their velocities are v_{1f} and v_{2f} as shown in the figure:

As the collision is elastic, then by law of conservation of linear momentum, we have:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad \text{----------} \quad (1)$$



$$m_1 v_{1i} - m_1 v_{1f} = m_2 v_{2f} - m_2 v_{2i}$$

$$m_1\left(v_{1i} - v_{1f}\right) = m_2\left(v_{2f} - v_{2i}\right) \quad \text{----------} \quad (2)$$

From the conservation of kinetic energy:

$$\frac{1}{2} m_1 v_{1i}^{\ 2} + \frac{1}{2} m_2 v_{2i}^{\ 2} = \frac{1}{2} m_1 v_{1f}^{\ 2} + \frac{1}{2} m_2 v_{2f}^{\ 2}$$

$$m_1 v_{1i}^{\ 2} + m_2 v_{2i}^{\ 2} = m_1 v_{1f}^{\ 2} + m_2 v_{2f}^{\ 2}$$

$$m_1 v_{1i}^{\ 2} - m_1 v_{1f}^{\ 2} = m_2 v_{2f}^{\ 2} - m_2 v_{2i}^{\ 2}$$

$$m_1\left(v_{1i}^{\ 2} - v_{1f}^{\ 2}\right) = m_2\left(v_{2f}^{\ 2} - v_{2i}^{\ 2}\right)$$

$$m_1\left(v_{1i} + v_{1f}\right)\left(v_{1i} - v_{1f}\right) = m_2\left(v_{2f} + v_{2i}\right)\left(v_{2f} - v_{2i}\right) \quad \text{-----------} \quad (3)$$

Dividing equations (2) and (3)

$$\frac{m_1\left(v_{1i} + v_{1f}\right)\left(v_{1i} - v_{1f}\right)}{m_1\left(v_{1i} - v_{1f}\right)} = \frac{m_2\left(v_{2f} + v_{2i}\right)\left(v_{2f} - v_{2i}\right)}{m_2\left(v_{2f} - v_{2i}\right)}$$

$$\left(v_{1i} + v_{1f}\right) = \left(v_{2f} + v_{2i}\right) \quad \text{----------} \quad (4)$$

$$v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

$$\left(v_{1i} - v_{2i}\right) = -\left(v_{1f} - v_{2f}\right)$$

$$v_{rel,i} = -v_{rel,f}$$

So, the relative velocity of approach before collision is equal and opposite to the relative velocity of separation after collision. From equation (4), we have:

$$v_{2f} = v_{1i} + v_{1f} - v_{2i} \quad \text{----------} \quad (5)$$

Putting the value of v_{2f} in equation (1)

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2\left(v_{1i} + v_{1f} - v_{2i}\right)$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{1i} + m_2 v_{1f} - m_2 v_{2i}$$

$$m_1 v_{1i} + m_2 v_{2i} + m_2 v_{2i} - m_2 v_{1i} = m_1 v_{1f} + m_2 v_{1f}$$

$$\left(m_1 - m_2\right) v_{1i} + 2 m_2 v_{2i} = \left(m_1 + m_2\right) v_{1f}$$

$$\left(m_1 + m_2\right) v_{1f} = \left(m_1 - m_2\right) v_{1i} + 2 m_2 v_{2i}$$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2 m_2}{m_1 + m_2}\right) v_{2i} \quad \text{----------} \quad (a)$$

Putting this value of v_{1f} in equation (5)

$$v_{2f} = v_{1i} + \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2 m_2}{m_1 + m_2}\right) v_{2i} - v_{2i}$$

$$v_{2f} = \left(1 + \frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2} - 1\right) v_{2i}$$

$$v_{2f} = \left(\frac{m_1 + m_2 + m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2 - (m_1 + m_2)}{m_1 + m_2}\right) v_{2i}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i} \quad \text{----------} \quad \text{(b)}$$

Special Cases:

Case 1. Equal Masses

When the colliding balls have equal mass, then by putting $m_1 = m_2$ in equations (a) and (b), we have:

$$v_{1f} = v_{2i}$$

$$v_{2f} = v_{1i}$$

Thus, if the balls of same masses collies each other, they will interchange their velocities after collision.

Case 2. Equal Masses and Target Particle at Rest

Putting $m_1 = m_2$, $v_{2i} = 0$ in equation (a) and (b), we have:

$$v_{1f} = 0$$

$$v_{2f} = v_{1i}$$

Thus, the ball of mass m_1, after collision, will come to stop and m_2 will takes of the velocity of m_1.

Case 3: When a Light Body Collides with the Massive Body at Rest.

When the mass of the target is much greater than that of incident particle i.e., $m_2 \gg m_1$, then we have $m_1 \approx 0$ (m_1 is negligible). For the present case, putting $m_1 = 0$ and $v_{2i} = 0$ in equations (a) and (b):

$$v_{1f} = -v_{1i}$$

$v_{2f} = 0$ Thus, the body of mass m_1 will bounce back with the same velocity while m_2 will remain stationary.

Case 4: When a Massive Body Collides with the Light Stationary Body.

When the incident particle is much massive as compared to the target $m_1 \gg m_2$, then we have $m_2 \approx 0$ (m_2 is negligible). For the present case, putting $m_2 = 0$ and $v_{2i} = 0$ in equations (a) and (b):

$$v_{1f} = v_{1i}$$

$$v_{2f} = 2v_{1i}$$

Hence the massive body will move with same velocity, while the lighter target at rest moves at twice the speed of massive projectile.

Inelastic Collision

Consider an inelastic collision in which the kinetic energy is not conserved, however momentum is conserved.

Let in a completely inelastic collision, both particles stick together after collision and move with common velocity v_f. So there is only on unknown v_f and the momentum equation is sufficient.

$$m_1 v_{1i} + m_2 v_{i2} = \left(m_1 + m_2\right) v_f$$

$$\Rightarrow v_f = \left(\frac{m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2}{m_1 + m_2}\right) v_{2i} \quad \text{----------} \quad (1)$$

When m_2 is initially at rest i.e., $v_{2i} = 0$, then

$$v_f = \left(\frac{m_1}{m_1 + m_2}\right) v_{1i}$$

The equation (1) can also be applied equally well in reverse process. So that a particle of mass M moving with velocity V splits up into two particles m_1 and $m_2 (= M - m_1)$ moving in opposite direction. So,

$$MV = m_1 v_{1i} + m_2 v_{i2} \quad \text{----------} \quad (2)$$

Special case:

If the original particle is at rest i.e., $V = 0$, then equation (2) will become:

$$0 = m_1 v_{1i} + m_2 v_{i2}$$

$$\frac{v_{1i}}{v_{i2}} = -\frac{m_2}{m_1}$$

So, the more massive particle moves with smaller velocity and vice versa. The two particles move in opposite directions.

Sample Problem 2.

By what fraction is the kinetic energy of neutron (mass m_1) decreased in a head on elastic collision with an atomic nucleus (mass m_2) initially at rest? Find the fractional decrease in kinetic energy of a neutron when it collides in this way with a lead nucleus, a carbon nucleus and a hydrogen nucleus. The ratio of nuclear mass to neutron mass $\left(\frac{m_2}{m_1}\right)$ is 206 for lead, 12 for carbon and 1 for hydrogen.

Solution.

Initial KE of neutron $K_i = \frac{1}{2} m_1 v_{1i}^2$

Final KE of neutron $K_f = \frac{1}{2} m_1 v_{1f}^2$

Fractional decrease in KE:

$$\frac{K_i - K_f}{K_i} = \frac{\frac{1}{2} m_1 v_{1i}^2 - \frac{1}{2} m_1 v_{1f}^2}{\frac{1}{2} m_1 v_{1i}^2}$$

$$\Rightarrow \frac{K_i - K_f}{K_i} = \frac{v_{1i}^2 - v_{1f}^2}{v_{1i}^2} = 1 - \frac{v_{1f}^2}{v_{1i}^2} \quad \text{----------} \quad (1)$$

The velocity of neutron after collision with massive target at rest:

$$v_f = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i}$$

$$\Rightarrow \frac{v_f}{v_{1i}} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)$$

$$\Rightarrow \frac{v_f^2}{v_i^2} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2$$

Putting value in equation (1), we have

$$\frac{K_i - K_f}{K_i} = 1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2$$

$$\Rightarrow \frac{K_i - K_f}{K_i} = 1 - \frac{m_1^2 + m_2^2 - 2m_1m_2}{m_1^2 + m_2^2 + 2m_1m_2}$$

$$\Rightarrow \frac{K_i - K_f}{K_i} = \frac{m_1^2 + m_2^2 + 2m_1m_2 - \left(m_1^2 + m_2^2 - 2m_1m_2\right)}{m_1^2 + m_2^2 + 2m_1m_2}$$

$$\Rightarrow \frac{K_i - K_f}{K_i} = \frac{m_1^2 + m_2^2 + 2m_1m_2 - m_1^2 - m_2^2 + 2m_1m_2}{\left(m_1 + m_2\right)^2}$$

$$\Rightarrow \frac{K_i - K_f}{K_i} = \frac{4m_1m_2}{\left(m_1 + m_2\right)^2}$$

(b) **For lead** $\frac{m_2}{m_1} = 206 \Rightarrow m_2 = 206m_1$

$$\frac{K_i - K_f}{K_i} = \frac{4m_1\left(206m_1\right)}{\left(m_1 + 206m_1\right)^2} = \frac{4(206)m_1^2}{\left(207m_1\right)^2}$$

$$\frac{K_i - K_f}{K_i} = 0.02 = 2\%$$

For carbon $\frac{m_2}{m_1} = 12 \Rightarrow m_2 = 12m_1$

$$\frac{K_i - K_f}{K_i} = \frac{4m_1\left(12m_1\right)}{\left(m_1 + 12m_1\right)^2} = \frac{4(12)m_1^2}{\left(13m_1\right)^2}$$

$$\frac{K_i - K_f}{K_i} = 0.28 = 28\%$$

For hydrogen $\frac{m_2}{m_1} = 1 \Rightarrow m_2 = 1m_1$

$$\frac{K_i - K_f}{K_i} = \frac{4m_1\left(1m_1\right)}{\left(m_1 + 1m_1\right)^2} = \frac{4(1)m_1^2}{\left(2m_1\right)^2}$$

$$\frac{K_i - K_f}{K_i} = 1 = 100\%$$

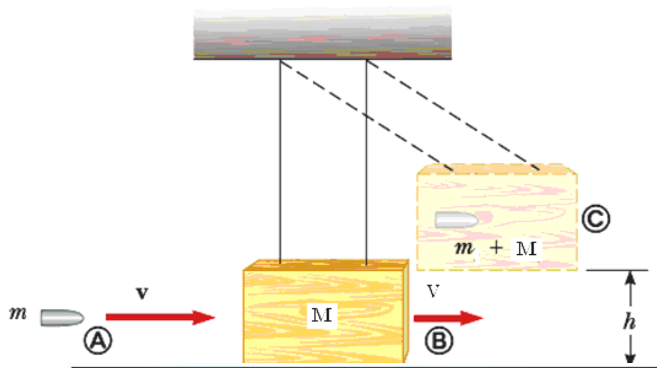
So, the material containing high content of hydrogen such as paraffix or water would be good moderation in a nuclear reactor.

Sample Problem 3.

A ballistic pendulum is a device that as used to measure the speeds of bullets before electronic timing devices were available. It consists of a large block of wood of mass M hanging from two long pairs of cords. A bullet of mass m is fired into the block and comes quickly to rest relative to the block. The block + bullet combination swings upwards, its center of mass rising a vertical distance h before the pendulum comes momentarily to rest at the end of its arc. Take the mass of the block to be M = 5.4 kg and the mass of the bullet to be 9.5 g.

(a) What is the initial speed of the bullet if the block rises to a height of h = 6.3 cm? (b) What is the initial K.E of the bullet? How much of this energy remains as mechanical energy of swing pendulum.

Solution.



The bullet and the block stick together in an inelastic collision. By law of conservation of momentum:

$mv + M(0) = (m+M)V$ ---------(1)

Where

$v =$ Velocity of the bullet before impact

$V =$ Velocity of combination (bullet + block) after impact

Mechanical (kinetic) energy is not conserved during the collision of bullet with block, it is conserved in the swinging pendulum after the impact.

K.E. of the combination at bottom = P. E. of combination at height h

$$\frac{1}{2}(M+m)V^2 = (M+m)gh$$

$$\frac{1}{2}V^2 = gh \quad \text{----------} \quad (2)$$

From equation (1) $V = \dfrac{mv}{(m+M)}$, put in equation (2):

$$\frac{1}{2}\left(\frac{mv}{m+M}\right)^2 = gh$$

$$\frac{1}{2}\left(\frac{m}{m+M}\right)^2 v^2 = gh$$

$$v^2 = 2\left(\frac{m+M}{m}\right)^2 gh$$

$$v = \left(\frac{m+M}{m}\right)^2 \sqrt{2gh}$$

$$v = \left(\frac{5.4+0.0095}{0.0095}\right)^2 \sqrt{2(9.8)(0.63)}$$

$$v = 630\,ms^{-1}$$

(b) Initial K.E. of bullet

$$K_b = \frac{1}{2}mv^2 = \frac{1}{2}(0.0095)(630)^2$$

$$K_b = 1900\,J$$

$$\begin{Bmatrix}\text{The mechanical energy of the}\\ \text{swinging pendulum}\end{Bmatrix} = \begin{Bmatrix}\text{P.E. of swinging pendulum}\\ \text{at the top of its swing}\end{Bmatrix}$$

$$E = (M+m)gh$$

$$E = (5.4+0.0095)(9.8)(0.063)$$

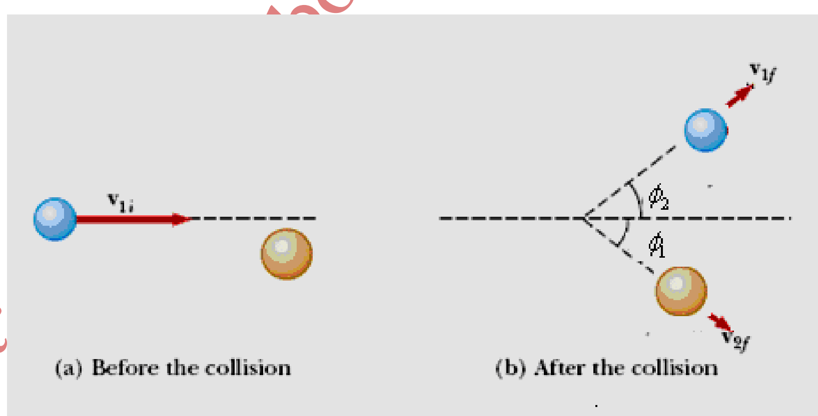
$$E = 3.3\,J$$

So only $\frac{3.3}{1900}$ or 0.2% of the original K.E of the bullet is transferred to mechanical energy of the pendulum. The rest energy is stored insider the pendulum block as the internal energy or transferred to environment, for example as heat and sound.

Two Dimensional Collision (Elastic)

When the two objects don't collide head on manner, then they move in different direction which doesn’t coincide with the original direction of motion. This is called oblique collision. Consider an incident particle (projectile) of mass m_1 moves with velocity v_{1i} in positive x-direction towards a particle (target particle) of mass m_2 initially at rest i.e., $v_{2i} = 0$.

The perpendicular distance ‘b’ between the line of motion incident particle and a line through m_2 is called the impact parameter. If $b = 0$, it is head on collision. If $b > 0$, the collision is oblique or glancing.



(a) Before the collision (b) After the collision

Figure shows the path of two objects. The direction of velocity vectors v_{1f} and v_{2f} are shown by ϕ_1 and ϕ_2 respectively.

Applying the law of conservation of linear momentum.

For x-component;

$$p_{ix} = p_{fy}$$

$$m_1 v_{1i} + 0 = m_1 v_{1f}\cos\phi_1 + m_2 v_{2f}\cos\phi_2 \quad ---------- \quad (1)$$

For y-component;

$$p_{iy} = p_{fy}$$

$$0+0=-m_1v_{1f}\sin\phi_1+m_2v_{2f}\sin\phi_2 \quad \text{----------} \quad (2)$$

If the collision is elastic, then by law of conservation of KE

$$\frac{1}{2}m_1v_{1i}^2+0=\frac{1}{2}m_1v_{1f}^2+\frac{1}{2}m_2v_{2f}^2 \quad \text{----------} \quad (3)$$

Here known quantities are m_1, m_2 *and* v_{1i}. And the unknown quantities are v_{1f}, v_{2f}, ϕ_1 and ϕ_2. But only three equations are there, which cannot determine there four unknowns. So, by applying an additional restrictions; say a particular angle θ, we can find the remaining three unknowns from three equations (1), (2) and (3).

Sample Problem 4.

A gas molecule having a speed of 322 ms^{-1} collides elastically with another molecule of the same mass which is initially at rest. After the collision, the first molecule moves at an angle of 30º to its initial direction. Find the speed of each molecule after collision and the angle made with the incident direction by the recoiling target molecule.

Solution.

$$m_1=m_2$$

$$v_{1i}=322\,ms^{-1}$$

$$v_{2i}=0$$

$$\phi_1=30^\circ$$

By the law of conservation of momentum: Along x-axis

$$m_1v_{1i}+0=m_1v_{1f}\cos\phi_1+m_2v_{2f}\cos\phi_2$$

Putting $m_1=m_2$ we have:

$$v_{1i}=v_{1f}\cos\phi_1+v_{2f}\cos\phi_2 \quad \text{----------} \quad (1)$$

By the law of conservation of momentum: Along y-axis

$$0+0=-m_1v_{1f}\sin\phi_1+m_2v_{2f}\sin\phi_2$$

Putting $m_1=m_2$ we have:

$$v_{1f}\sin\phi_1=v_{2f}\sin\phi_2 \quad \text{----------} \quad (2)$$

By law of conservation of KE

$$\frac{1}{2}m_1v_{1i}^2+0=\frac{1}{2}m_1v_{1f}^2+\frac{1}{2}m_2v_{2f}^2$$

Putting $m_1=m_2$ we have:

$$v_{1i}^2=v_{1f}^2+v_{2f}^2 \quad \text{----------} \quad (3)$$

Now we eliminate ϕ_2. By equation (1):

$$v_{1i}-v_{1f}\cos\phi_1=v_{2f}\cos\phi_2 \quad \text{----------} \quad (4)$$

Squaring and adding equation (2) and (4):

$$\left(v_{1i}-v_{1f}\cos\phi_1\right)^2+v_{1f}^2\sin^2\phi_1=v_{2f}^2\cos^2\phi_2+v_{2f}^2\sin^2\phi_2$$

$$v_{1i}^2 + v_{1f}^2 \cos^2\phi_1 - 2v_{1i}v_{1f}\cos\phi_1 + v_{1f}^2 \sin^2\phi_1 = v_{2f}^2\left(\cos^2\phi_2 + \sin^2\phi_2\right)$$

$$v_{1i}^2 + v_{1f}^2\left(\cos^2\phi_1 + \sin^2\phi\right) - 2v_{1i}v_{1f}\cos\phi_1 = v_{2f}^2$$

$$v_{1i}^2 + v_{1f}^2 - 2v_{1i}v_{1f}\cos\phi_1 = v_{2f}^2 \quad \text{----------} \quad (5)$$

From equation (3):

$$v_{1i}^2 - v_{1f}^2 = v_{2f}^2 \quad \text{----------} \quad (6)$$

Comparing equations (5) and (6):

$$v_{1i}^2 + v_{1f}^2 - 2v_{1i}v_{1f}\cos\phi_1 = v_{1i}^2 - v_{1f}^2$$

$$v_{1f}^2 - 2v_{1i}v_{1f}\cos\phi_1 = -v_{1f}^2$$

$$2v_{1f}^2 = 2v_{1i}v_{1f}\cos\phi_1$$

$$v_{1f} = v_{1i}\cos\phi_1$$

$$v_{1f} = (322)\cos 30^o$$

$$v_{1f} = 270\,ms^{-1}$$

Putting this value in equation (3), we have

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2$$

$$v_{2f} = \sqrt{(322)^2 - (279)^2} = 161\,ms^{-1}$$

From equation (2):

$$v_{1f}\sin\phi_1 = v_{2f}\sin\phi_2$$

$$\sin\phi_2 = \frac{v_{1f}}{v_{2f}}\sin\phi_1$$

$$\sin\phi_2 = \frac{279}{161}\sin 30^o = 0.866$$

$$\phi_2 = \sin^{-1}(0.866)$$

$$\phi_2 = 60^o$$

Inelastic Collision in Two Dimensions

Consider a completely inelastic collision in which two bodies of masses m_1 and m_2 moves with velocities v_{1i} and v_{2i} collide with each other. These bodies meet and stick together at origin