SYSTEM OF PARTICLES

Two Particle System

Center of Mass

It is the point at which the whole mass of an object or system of particles may suppose to be concentrated.

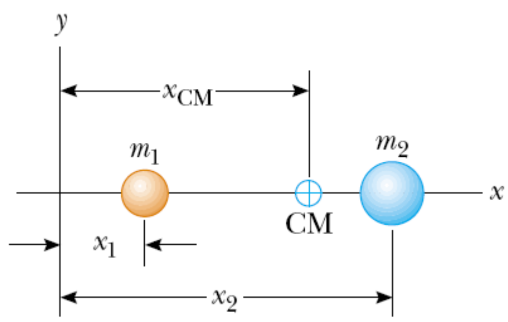
This point is easy to locate in perfectly symmetrical objects because it coincides with the center of symmetry. For example, in a sphere it is center of sphere and in case of a square, it is the center of square. For irregular shapes and system of particles, mathematical methods are necessary to find the center of mass.

Explanation

Consider a system of two particles of mass m_1 and m_2 having displacements x_1 and x_2 along x-axis respectively.

Let M is the total mass of the system $(M = m_1 + m_2)$ which may be concentrated at point C, whose displacement x_{CM}.

The position of the center of mass is defined as:



$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{M}$$

$$x_{CM} = \frac{1}{M}(m_1 x_1 + m_2 x_2) \quad \text{----------} \quad (1)$$

The center of mass of the system of two particles may not be a point on the either bodies.

Velocity of Center of Mass

The velocity of center of mass v_{CM} is taken by the time derivative of equation (1):

$$v_{CM} = \frac{dx_{CM}}{dt}$$

$$v_{CM} = \frac{d}{dt}\left[\frac{1}{M}(m_1 x_1 + m_2 x_2)\right]$$

$$v_{CM} = \frac{1}{M}\left(m_1 \frac{dx_1}{dt} + m_2 \frac{dx_2}{dt}\right)$$

$$v_{CM} = \frac{1}{M}(m_1 v_1 + m_2 v_2) \quad \text{----------} \quad (2)$$

Where v_1 and v_2 are the velocities of bodies of masses m_1 and m_2, respectively.

Acceleration of Center of Mass

The acceleration of center of mass a_{CM} is taken by the time derivative of equation (2):

$$a_{CM} = \frac{dv_{CM}}{dt}$$

$$a_{CM} = \frac{d}{dt}\left[\frac{1}{M}(m_1 v_1 + m_2 v_2)\right]$$

$$a_{CM} = \frac{1}{M}\left(m_1\frac{dv_1}{dt} + m_2\frac{dv_2}{dt}\right)$$

$$a_{CM} = \frac{1}{M}(m_1 a_1 + m_2 a_2) \quad \text{----------} \quad (3)$$

Where a_1 and a_2 are the acceleration of bodies of masses m_1 and m_2, respectively.

Case 1. External Forces are not Present

Let the masses m_1 and m_2 are interacting with one another and no external forces are applied on the system.

Let F_{12} is the force exerted on m_1 by m_2

F_{21} is the force exerted on m_2 by m_1

So $F_{12} = m_1 a_1$

$F_{21} = m_2 a_2$

As the bodies are interacting with each other, then by Newton's third law of motion:

$$F_{12} = -F_{21} \quad \text{----------} \quad (4)$$

By equation (3):

$$a_{CM} = \frac{1}{M}(F_{12} + F_{21})$$

Using equation (4):

$$a_{CM} = \frac{1}{M}(F_{12} - F_{12})$$

$$\Rightarrow a_{CM} = \frac{1}{M}(0)$$

$$\Rightarrow a_{CM} = 0$$

It means that if there is no external force acting on the system, then the acceleration of center of mass is zero. So, the center of mass moves with constant velocity in the absence of any external force.

Case 2. External Forces are Present

Let F_{ext1} and F_{ext2} are the external forces acting on the masses m_1 and m_2 respectively, in addition to internal forces F_{12} and F_{21}. So the net force (sum of internal and external forces) on m_1 is

$$F_{ext1}+F_{12}=m_1a_1 \quad ---------- \quad (5)$$

Similarly, the net force (sum of internal and external forces) on m_2 is

$$F_{ext2}+F_{21}=m_2a_2 \quad ---------- \quad (6)$$

From equation (3), we have:

$$Ma_{CM}=m_1a_1+m_2a_2$$

Putting values from equation (5) and equation (6):

$$Ma_{CM}=\left(F_{ext1}+F_{12}\right)+\left(F_{ext2}+F_{21}\right)$$

$$\Rightarrow Ma_{CM}=\left(F_{ext1}+F_{12}\right)+\left(F_{ext2}-F_{12}\right) \qquad \text{Using equation (4)}$$

$$\Rightarrow Ma_{CM}=F_{ext1}+F_{12}+F_{ext2}-F_{12}$$

$$\Rightarrow Ma_{CM}=F_{ext1}+F_{ext2}$$

$$\Rightarrow Ma_{CM}=\sum F_{ext}$$

So we conclude that the net external force $\sum F_{ext}$ is acting on the center of mass, where whole mass of the system is concentrated.

Many Particle System

Now we generalize our case from two particle system to many particles system.

Let us consider a system of N-particles of masses $m_1,m_2,m_3,....,m_n$ having position vectors $\vec{r}_1(x_1,y_1,z_1),\vec{r}_2(x_2,y_2,z_2),....,\vec{r}_n(x_n,y_n,z_n)$ with respect to the origin of rectangular coordinate system.

The total mass of the system M is:

$$M=m_1+m_2+m_3+....+m_n$$

Let the total mass of the system is concentrated at the center of mass of the system having position vector $r_{CM}(x_{CM},y_{CM},z_{CM})$, where

$$x_{CM}=\frac{m_1x_1+m_2x_2+m_3x_3+....+m_nx_n}{m_1+m_2+m_3+....+m_n}$$

$$\Rightarrow x_{CM}=\frac{m_1x_1+m_2x_2+m_3x_3+....+m_nx_n}{M}$$

$$\Rightarrow x_{CM}=\frac{1}{M}\left(m_1x_1+m_2x_2+m_3x_3+....+m_nx_n\right)$$

$$\Rightarrow x_{CM}=\frac{1}{M}\sum_{i=1}^{n}m_ix_i \quad ---------- \quad (1)$$

Similarly,

$$y_{CM}=\frac{1}{M}\left(m_1y_1+m_2y_2+m_3y_3+....+m_ny_n\right)$$

$$\Rightarrow y_{CM}=\frac{1}{M}\sum_{i=1}^{n}m_iy_i \quad ---------- \quad (2)$$

Also,

$$z_{CM} = \frac{1}{M}\left(m_1 z_1 + m_2 z_2 + m_3 z_3 + ... + m_n z_n\right)$$

$$\Rightarrow z_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i \quad ---------- \quad (3)$$

Combining equation (1), (2) and (3), we have:

$$\vec{r}_{CM} = \frac{1}{M}\left(m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + ... + m_n \vec{r}_n\right)$$

$$\Rightarrow \vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i \quad ---------- \quad (4)$$

Velocity

The velocity of the center of mass of the system is given by:

Velocity of Center of Mass

The velocity of center of mass v_{CM} is taken by the time derivative of equation (1):

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt}$$

$$\Rightarrow \vec{v}_{CM} = \frac{d}{dt}\left[\frac{1}{M}\left(m_1 \vec{r}_1 + m_2 \vec{r}_2 + m_3 \vec{r}_3 + ... + m_n \vec{r}_n\right)\right]$$

$$\Rightarrow \vec{v}_{CM} = \frac{1}{M}\left(m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + m_3 \frac{d\vec{r}_3}{dt} + ... + m_n \frac{d\vec{r}_n}{dt}\right)$$

$$\Rightarrow \vec{v}_{CM} = \frac{1}{M}\left(m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3 + ... + m_n \vec{v}_n\right)$$

$$\Rightarrow \vec{v}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{v}_i \quad ---------- \quad (5)$$

Acceleration of Center of Mass

The acceleration of center of mass a_{CM} is taken by the time derivative of equation (2):

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt}$$

$$\Rightarrow \vec{a}_{CM} = \frac{d}{dt}\left[\frac{1}{M}\left(m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \dots + m_n\vec{v}_n\right)\right]$$

$$\Rightarrow \vec{a}_{CM} = \frac{1}{M}\left(m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + m_3\frac{d\vec{v}_3}{dt} + \dots + m_n\frac{d\vec{v}_n}{dt}\right)$$

$$\Rightarrow \vec{a}_{CM} = \frac{1}{M}\left(m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \dots + m_n\vec{a}_n\right)$$

$$\Rightarrow \vec{a}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{a}_i \qquad \text{----------} \quad (6)$$

The above expression can also be written as:

$$M\vec{a}_{CM} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \dots + m_n\vec{a}_n$$

$$\Rightarrow M\vec{a}_{CM} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots + \vec{F}_n$$

$$\Rightarrow M\vec{a}_{CM} = \sum_{i=1}^{n}\vec{F}_i$$

$$\Rightarrow M\vec{a}_{CM} = \vec{F}$$

Where $\sum_{i=1}^{n}\vec{F}_i = F$ is the net force acting on the system of particles.

This shows that the net force acting on system of particles is equal to the mass of the system times the acceleration f center of mass of system.

Internal and External Forces

Suppose there are internal as well as external forces acting on the particles. The internal force are due to interaction of the particles among themselves.

Let F_{nk} is the force applied on object of mass m_n by the particle of mass m_k

F_{kn} is the force applied on object of mass m_k by the particle of mass m_n

By Newton's third law,

$$F_{nk} = -F_{kn}$$

Or $F_{nk} + F_{kn} = 0$

As the internal forces acts in pairs (action-reaction pairs), so, they cancel each other. Hence only the external forces are the net forces acting on the system of particles. Thus,

$$\sum\vec{F}_{ext} = M\vec{a}_{CM}$$

In component form;

$$\sum\vec{F}_{ext,x} = M\vec{a}_{CM,x} \qquad \sum\vec{F}_{ext,y} = M\vec{a}_{CM,y} \qquad \sum\vec{F}_{ext,z} = M\vec{a}_{CM,z}$$

Conclusion

The overall translational motion of a system of particles can be analyzed using Newton's laws as if all the mass were concentrated at the centre of mass and the total external forces were applied at that point.

If the resultant of forces is zero, i.e.,

$$\sum \vec{F}_{ext} = 0$$

Then $\vec{a}_{CM} = 0$

So the center of mass moves with constant velocity.

Center of Mass of the Solid Objects

Suppose we have a solid object. Its center of mass cannot be easily found using the relation:

$$\vec{r}_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$$

We divide the solid into tiny elements each having mass δm. If each element becomes infinitely small i.e., $\delta m \to 0$, then

$$x_{CM} = \frac{1}{M}\lim_{\delta m \to 0}\sum_{i=1}^{n} x_i \delta m_i = \frac{1}{M}\int x\, dm$$

$$y_{CM} = \frac{1}{M}\lim_{\delta m \to 0}\sum_{i=1}^{n} y_i \delta m_i = \frac{1}{M}\int y\, dm$$

$$z_{CM} = \frac{1}{M}\lim_{\delta m \to 0}\sum_{i=1}^{n} z_i \delta m_i = \frac{1}{M}\int z\, dm$$

In vector form, these equations are written as:

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\, dm$$

The process of integration provides an easy to find the center of mass of a solid.

In terms of uniform volume mass density, the above equation becomes:

$$x_{CM} = \frac{1}{M}\int x\, \rho\, dv$$

$$y_{CM} = \frac{1}{M}\int y\, \rho\, dv$$

$$z_{CM} = \frac{1}{M}\int z\, \rho\, dv$$

In vector form,

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\, \rho\, dv$$

CALCULATION OF CENTER OF MASS OF VARIOUS OBJECTS

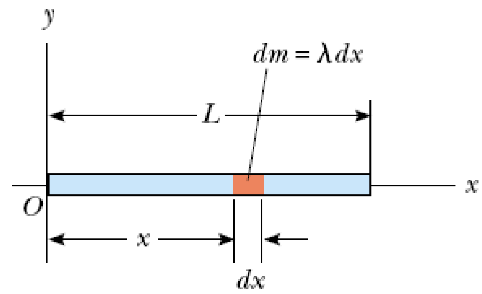
Calculation of Center of Mass of a Uniform Rod

Let us take a rod of length L arranged along x-axis. Since the rod is uniform, its mass is uniformly distributed along its length. We can exploit the concept of linear mass density defined by:

$$\mu = \lim_{\delta l \to 0} \frac{\Delta m}{\Delta l} = \frac{dm}{dl} = \frac{dm}{dx}$$

$$dm = \lambda dx$$

And $Total\ Mass\ \ M = \int dm = \int_0^L \lambda dx$



Obviously, the center of mass will lie somewhere along its length:

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{\int_0^L \lambda\, x\, dx}{\int_0^L \lambda dx} = \frac{\int_0^L x\, dx}{\int_0^L dx}$$

$$x_{CM} = \frac{\left|\frac{1}{2}x^2\right|_0^L}{\left|x\right|_0^L} = \frac{\frac{1}{2}L^2}{L} = \frac{1}{2}L$$

i.e., the center of mass of a uniform rod lies at its center.

Calculation of Center of Mass of a Uniform Solid Cylinder

Consider a uniform solid cylinder of length L arranged along the x-axis. Since the solid cylinder is uniform, its mass is uniformly distributed along its length.

The volume mass density ρ is described as:

$$\rho = \lim_{\delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

And $\quad M = \int dm = \int \rho\, dV$

If R is the radius of cylinder, then the volume of the element of the cylinder of infinitesimal length dx as $dV = \pi R^2 dx$.

Now

$$x_{CM} = \frac{1}{M}\int x\,dm = \frac{\int_0^L x\rho\, dV}{\int_0^L \rho dV} = \frac{\int_0^L x\rho\,\pi R^2 dx}{\int_0^L \rho\pi R^2 dx}$$

$$x_{CM} = \frac{\int_0^L x\, dx}{\int_0^L dx} = \frac{\left|\frac{1}{2}x^2\right|_0^L}{\left|x\right|_0^L} = \frac{\frac{1}{2}L^2}{L} = \frac{1}{2}L$$

Sample Problem 4:

A thin strip of material is bent into the shape of a semicircle of radius R. Find its center of mass.

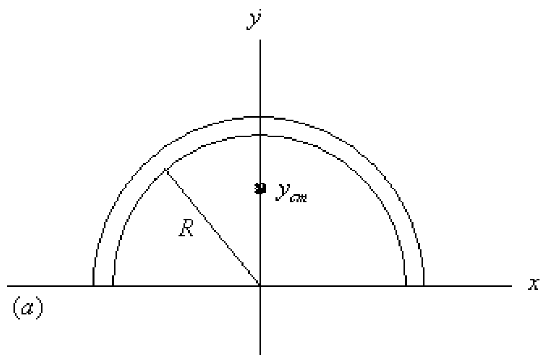
Solution.

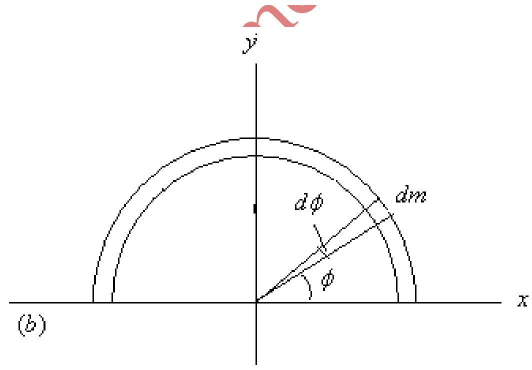
By symmetry we can see that the x-coordinate of center of mass is zero $x_{CM} = 0$ and center of mass of the object under consideration lies on y-axis.

Let the element of mass dm makes and angle $d\phi$. The total mass makes an angle π. So,

$$\frac{dm}{M} = \frac{d\phi}{\pi}$$

Or $$dm = \frac{M}{\pi} d\phi$$





The y-coordinate of element dm is located at $y = R\sin\phi$. The center of mass of this object is:

$$y_{CM} = \frac{1}{M}\int y\,dm$$

$$y_{CM} = \frac{1}{M}\int_0^{\pi} R\sin\phi \frac{M}{\pi} d\phi$$

$$y_{CM} = \frac{R}{\pi}\int_0^{\pi} \sin\phi\, d\phi$$

$$y_{CM} = -\frac{R}{\pi}\left|\cos\phi\right|_0^{\pi}$$

$$y_{CM} = -\frac{R}{\pi}\left[\cos\pi - \cos 0\right]$$

$$y_{CM} = -\frac{R}{\pi}\left[-1-1\right] = -\frac{R}{\pi}\left[-2\right]$$

$$y_{CM} = \frac{2}{\pi}R = 0.637\,R$$

Note that the center of mass does not need to be within the volume or the material of an object.

System of Variable Mass

Consider a system of variable mass and let

$M =$ Mass of the system moving with velocity v in the observer's FOR

$\Delta M =$ Change in mass of the system after time Δt.

If ΔM is positive, the mass of the system will increase. It is negative if the total mass of the system decreases, as in case of rocket it is the mass of ejected gases ($-\Delta M$).

$M + \Delta M =$ Mass of the system moving with velocity $v + \Delta v$ after a time $t + \Delta t$.

Written and composed by: Prof. Muhammad Ali Malik (M. Phil. Physics), Govt. Degree College, Naushera

$U =$ Velocity of the ejected mass w r t observer's frame of reference.

$F_{ext} =$ External force such as gravity or air drag on the system.

Initial Momentum $P_i = MV$

Final Momentum $P_f = (M + \Delta M)(V + \Delta V) + (-\Delta M)U$

Now

Change in Momentum $\Delta P = P_f - P_i$

$$\Delta P = (M + \Delta M)(V + \Delta V) + (-\Delta M)U - MV$$

$$\Delta P = MV + M.\Delta V + \Delta M.V + \Delta M.\Delta V - \Delta M.U - MV$$

Neglecting $\Delta M.\Delta V$, being small, we get:

$$\Delta P = M.\Delta V + \Delta M.V - \Delta M.U$$

By Newton's 2nd law, we have:

$$F_{ext} = \frac{dP}{dt}$$

$$F_{ext} = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t}$$

$$F_{ext} = \lim_{\Delta t \to 0} \left(\frac{M.\Delta V + \Delta M.V - \Delta M.U}{\Delta t} \right)$$

$$F_{ext} = \lim_{\Delta t \to 0} M.\frac{\Delta V}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta M}{\Delta t}.V - \lim_{\Delta t \to 0} \frac{\Delta M}{\Delta t}.U$$

$$F_{ext} = \lim_{\Delta t \to 0} M.\frac{\Delta V}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta M}{\Delta t}(V - U)$$

$$F_{ext} = M.\frac{dv}{dt} + \frac{dM}{dt}(V - U) \quad \text{----------} \quad (1)$$

Here $\frac{dv}{dt}$ is the acceleration of system produced as it ejects mass $(-\Delta M)$ with velocity U and $\frac{dM}{dt}$ is the rate of ejection of mass.

Equation (1) can be written as:

$$F_{ext} = M.\frac{dv}{dt} + V\frac{dM}{dt} - U\frac{dM}{dt}$$

$$F_{ext} = \frac{d}{dt}(MV) - U\frac{dM}{dt}$$

If the mass of the system is constant $\frac{dM}{dt} = 0$, then the above equation become the single particular form of Newton's 2nd law.

Equation (1) can be used to analyze the motion of the rocket. Take $U - V = V_{rel}$ which is the relative velocity of the ejected mass relative to the rocket. Equation (1) becomes:

$$F_{ext} = M\frac{dv}{dt} - \frac{dM}{dt}(U - V)$$

$$F_{ext} = M\frac{dv}{dt} - V_{rel}\frac{dM}{dt}$$

$$M\frac{dv}{dt} = F_{ext} - V_{rel}\frac{dM}{dt}$$

The term $V_{rel}\frac{dM}{dt}$ is called thrust of the rocket.

The Rocket Equation

Consider a rocket in distant space (out of the field of gravity of the earth), where external force is zero i.e., $F_{ext} = 0$.

Let $\frac{dV}{dt}$ is in positive direction as the rocket accelerates and V_{rel} is the magnitude of the exhaust velocity and is in the negative direction. So, the eqation of motion of a system of variable mass:

$$M\frac{dV}{dt} = F_{ext} - V_{rel}\frac{dM}{dt} \text{ reduces to}$$

$$M\frac{dV}{dt} = -V_{rel}\frac{dM}{dt}$$

$$dV = -V_{rel}\frac{dM}{M}$$

Let

$M =$ Total mass of rocket which is variable

$M_0 =$ Original mass of rocket plus fuel.

$m_b =$ Mass of the burnt out fuel

$M_f = M_0 - m_b =$ Final mass of the rocket

$V_i =$ Initial velocity of the rocket

$V_f =$ final velocity of the rocket after burning fuel of mass m_b

Integrating the above equation:

$$\int_{V_i}^{V_f} dV = -V_{rel}\int_{M_0}^{M_f}\frac{dM}{M}$$

$$V_f - V_i = -V_{rel}\left|\ln M\right|_{M_0}^{M_f}$$

$$V_f - V_i = -V_{rel}\left[\ln M_f - \ln M_0\right]$$

$$V_f - V_i = -V_{rel}\ln\frac{M_f}{M_0}$$

$$V_f - V_i = -V_{rel}\ln\frac{M_0 - m_b}{M_0}$$

$$V_f - V_i = V_{rel}\ln\frac{M_0}{M_0 - m_b}$$

If $V_i = 0$

$$V_f = V_{rel}\ln\frac{M_0}{M_f}$$

$$\frac{V_f}{V_{rel}} = \ln\frac{M_0}{M_f}$$

$$-\frac{V_f}{V_{rel}} = -\ln\frac{M_0}{M_f} = \ln\frac{M_f}{M_0}$$

$$\ln\frac{M_f}{M_0} = -\frac{V_f}{V_{rel}}$$

$$\frac{M_f}{M_0} = \exp\left(-\frac{V_f}{V_{rel}}\right)$$

$$M_f = M_0 \exp\left(-\frac{V_f}{V_{rel}}\right)$$

This is the equation of mass of the rocket.

Sample Problem 12. A rocket has a mass of 13600 kg when fueled on the launching pad. It is fired vertically upward and at burnt-out, has consumed an ejected 9100 kg of fuel. Gases are exhausted at the rate of 146 kgs^{-1} with a speed of 1520 ms^{-1}, relative to the rocket, both quantities being assumed to be constant while the fuel is burning. (a) What is the thrust? (b) If one can neglect all external forces, including gravity and air resistance, what will be the velocity of the rocket at burnt-out.

Solution:

$M_0 =$ Original mass of rocket plus fuel = 13600 kg

$m_b =$ Mass of the burnt out fuel = 9100 kg

$M =$ Total mass of rocket which is variable = 13600 – 9100 4500 kg

$\frac{dM}{dt} =$ Rate of fuel ejection =146 kgs^{-1}

$V_{rel} = 1520$ ms^{-1}

$F_{thrust} = ?$

Upward Thrust $F_{thrust} = V_{rel}\frac{dM}{dt}$

$F_{thrust} = 1520 * 146 = 2.22 * 10^5\ N$

$v_f = ?$

$$V_f = V_{rel}\ln\left(\frac{M_0}{M_0 - m_b}\right)$$

$$V_f = 1520 * \ln\left(\frac{13600}{4500}\right)$$

$$V_f = 1680\,ms^{-1}$$