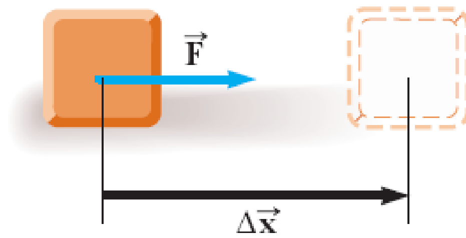
WORK AND ENERGY

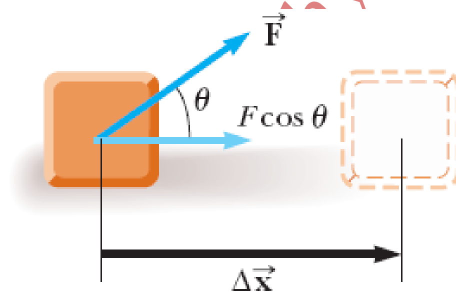
Work Done by the Constant Force

Consider a constant force F acts on a body and displaces it through a distance S in its own direction. Then the work done is defined as the product of magnitude of force and displacement:



Work $W = |\vec{F}||\Delta\vec{x}| = F\Delta x$

However if the force makes an angle θ with the direction of motion of the body, then work is defined as "the product of component of force along the line of motion and the magnitude of displacement". In this case, work is also defined as "the product of magnitude of force and the component of displacement along the direction of force. Thus



$$W = (F\cos\theta)\Delta x$$

Or $$W = F(\Delta x\cos\theta)$$

$$W = F\Delta x\cos\theta = \vec{F}.\Delta\vec{x}$$

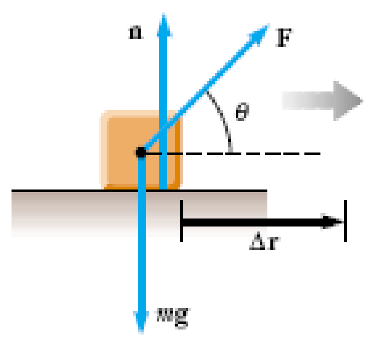
So, work is defined as the "Scalar product of force and displacement".

(a) (b) (c)

If there are number of forces acting on the body, then work done is separately calculated for each force. Then the net work will be the sum of work done by all separate forces.

<u>**Case 1.**</u> If $\theta = 0^\circ$

If the displacement is produced in the direction of the force i.e., $\theta = 0^\circ$, then work done will be:



$$W = \vec{F}.\Delta\vec{x}$$

$$W = F\Delta x\cos\theta = F\Delta x\cos 0^\circ$$

As $\cos 0^\circ = 1$

$$W = F\Delta x$$

Thus, when a horizontal force moves a body horizontally or when a vertical force lifts a body vertically, then the work done is the product of force and distance covered by the object.

<u>Case 2</u>. If $\theta = 90^\circ$

When the force has no componet in the direction of motion, then no work is done by the force.

$$W = \vec{F}.\Delta\vec{x}$$

$$W = F\Delta x\cos\theta = F\Delta x\cos 90^\circ$$

As $\cos 90^\circ = 0$

$$W = 0$$

Thus, if a person carrying a weight walks horizontally, then the force exerted by the man is perpendicular to the horizontal displacement. So, no work is done.

The examples of forces which don’t work are:

- Centripetal force
- Tension in the string of vibrating pendulum
- Weight and normal force don't work because they are perpendicular to the displacement.

<u>Case 3</u>. If $\theta = 180^\circ$

When the force has a component opposite to the direction of displacement, then the work done by the force

$$W = \vec{F}.\Delta\vec{x}$$

$$W = F\Delta x\cos\theta = F\Delta x\cos 180^\circ$$

As $\cos 180^\circ = -1$

$$W = -F\Delta x$$

The force of friction is opposite to the direction of motion, so the work done by the force of friction on the object is zero.

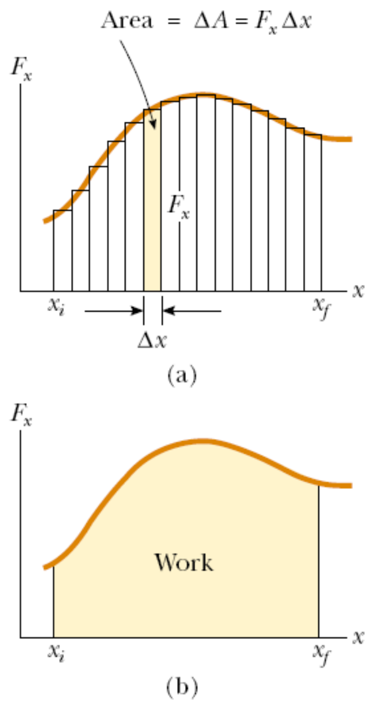
Joule

The work done is a scalar quantity and its unit is Joule. Work is one joule when force of one Newton acts on a body and displaces it through a distance of 1 meter in its own direction.

Note: The force is invariant and is independent of the choice of frame of reference. However, displacement is not invariant and depends upon the frame of reference, in which the measurements of displacement are carried out.

Work Done by a Variable Force (In One Dimension)

Let the force is in x-direction and its magnitude changes with respect to position. The variation of force with displacement x is described in the figure. Figure a shows the force is the function of displacement.



We have to calculate the work done from initial position x_i to final position x_f .

For this we divide the displacement into N small intervals each of length Δx .

Let F_1 is the magnitude of force during the first interval, then the work done by the force during the first interval is approximately written as:

$$\Delta W_1 = F_1 \Delta x$$

This is the shaded area of first rectangle. Similarly for other intervals:

$$\Delta W_2 = F_2 \Delta x$$

$$\Delta W_3 = F_3 \Delta x$$

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$$\Delta W_n = F_n \Delta x$$

The total work done from position x_i to position x_f is:

$$W = \Delta W_1 + \Delta W_2 + \Delta W_3 + ... + \Delta W_n$$

$$W = F_1 \Delta x + F_2 \Delta x + F_3 \Delta x + ... + F_n \Delta x$$

$$W = \sum_{i=1}^{N} F_i \Delta x$$

In order to obtain better results, we divide the displacement into a very large number of equal intervals such that $N \to \infty$ and $\Delta x \to 0$. Hence

$$W = \lim_{N \to \infty} \sum_{i=1}^{N} F_i \Delta x = \lim_{\Delta x \to 0} \sum_{i=1}^{N} F_i \Delta x$$

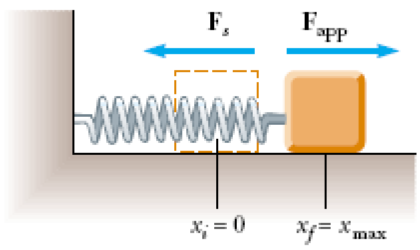
$$W = \int_{x_i}^{x_f} F \, \Delta x$$

Which is area under the curve.

Explanation (Work Done by Variable Force)

A good example of one dimensional force is in the case of mass-spring system as shown in the figure. External force $\vec{F}_{app}$ is applied to the body towards right, then

$$\vec{F}_{app} = k\vec{x}$$

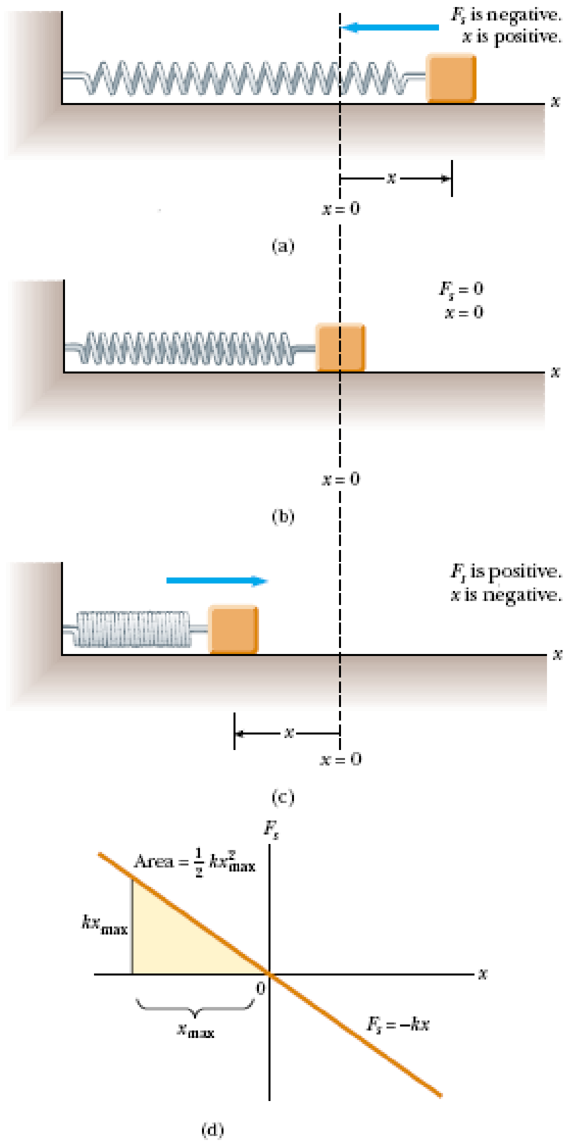


Where k is the spring constant and is the measure of stiffness or softness of the spring. x is the displacement in the of object from equilibrium position.

The restoring force F_S exerted by the spring on the body is:

$$F_S = -kx$$

This is known as the Hook's law.



Work done by the spring on the body, when the body moves from its initial position x_i to x_f will be:

$$W_S = \int_{x_i}^{x_f} F_S(x).dx$$

$$\Rightarrow W_S = \int_{x_i}^{x_f} (-kx)dx = \left| -\frac{1}{2}kx^2 \right|_{x_i}^{x_f}$$

$$\Rightarrow W_S = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

When the spring is stretched $(x_i < x_f)$, work is negative.

When the spring is compressed $(x_i > x_f)$, work is positive.

If the particle is moved from mean position $(x = 0)$ i.e., $x_i = 0$ to $x_f = x$, then the work done by the spring on the body will be:

$$W_S = \frac{1}{2}k(0)^2 - \frac{1}{2}kx^2$$

$$W_S = -\frac{1}{2}kx^2$$

Now the work done by the external force is:

$$W_{ext} = \int_0^x F_{ext}(x)dx = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

Work Done by a Variable Force (Two Dimensional Case)

In case of a variable force, its magnitude and direction may change so that the particle may move along curved path.

To find out the work done, we divide the path into a large number of small displacements δs. Each δs is along tangent to the path and points in the direction of motion. Work done during displacement δs_p is

$$\delta W_p = \vec{F}_p.\delta\vec{s}_p = F_p \cos\theta_p \ \delta s_p$$

The total work done between the points i and f is calculated by adding the elements of work for each segment δW_i:

$$W = \sum_{p=i}^{f} \vec{F}_p.\delta\vec{s}_p = \sum_{p=i}^{f} F_p \cos\theta_p \delta s_p$$

If δs becomes infinitesimally small such that $\delta s \to 0$, then δs can be replaced by dx and summation by integration:

$$W = \int_i^f \vec{F}.d\vec{s} = \int_i^f F\cos\theta ds \quad \text{----------} \quad (1)$$

As both F and θ vary point to point, so F will have non-zero horizontal as well as vertical components. Thus

$F = F_x\hat{i} + F_y\hat{j}$ and $d\vec{s} = dx\hat{i} + dy\hat{j}$

$$W = \int_i^f \vec{F}.d\vec{s}$$

$$W = \int_i^f \left(F_x\hat{i} + F_y\hat{j}\right).\left(dx\hat{i} + dy\hat{j}\right)$$

$$W = \int_i^f F_x dx + F_y dy \quad \text{----------} \quad (2)$$

Equation (1) and (2) are known as the line integral. In case of three dimensions:

$$W = \int_i^f F_x dx + F_y dy + F_z dz$$

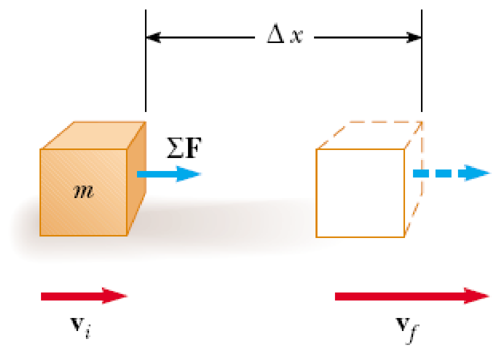
Work-Energy Theorem

Statement:

The net work done by the forces acting on the particle is equal to the change in the kinetic energy of the particle.

Derivation

When a net force acts on the body, it changes its state of motion by producing acceleration $\vec{a}$ in it. Let the net force F_{net} changes the velocity of the body from x_i to x_f, then



$$W_{net} = F_{net}(x_f - x_i)$$

$$W_{net} = ma.(x_f - x_i) \quad \text{----------} \quad (1)$$

$$F_{net} = ma\,(Newton\,Second\,Law)$$

Using 3rd equation of motion:

$$v_f^2 - v_i^2 = 2a(x_f - x_i)$$

$$a = \frac{\left(v_f^2 - v_i^2\right)}{2\left(x_f - x_i\right)}$$

Putting values in equation (1), we have:

$$W_{net} = m.\frac{\left(v_f^2 - v_i^2\right)}{2\left(x_f - x_i\right)}.(x_f - x_i)$$

$$W_{net} = m.\frac{\left(v_f^2 - v_i^2\right)}{2} = \frac{1}{2}m\left(v_f^2 - v_i^2\right)$$

$$W_{net} = m.\frac{\left(v_f^2 - v_i^2\right)}{2} = \frac{1}{2}m\left(v_f^2 - v_i^2\right)$$

$$W_{net} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

$$W_{net} = K.E_f - K.E_i = \Delta K.E$$

This is the mathematical form of work-energy theorem. This proof corresponds to the constant force acting on the o object. However, this expression is not valid for the case of non constant force.

General Proof of Work Energy Theorem

If a non-constant force acts on the object in one dimension, then the word done by the force on the object can be find out by using expression:

$$W_{net} = \int F_{net}\,dx \quad \text{----------} \quad (1)$$

Now $F_{net} = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = m\frac{dv}{dx}v = mv\frac{dv}{dx}$, thus equation (1) becomes:

$$W_{net} = \int mv\frac{dv}{dx}dx$$

$$W_{net} = \int_{v_i}^{v_f} mv\,dv$$

$$W_{net} = m\left|\frac{v^2}{2}\right|_{v_i}^{v_f}$$

$$W_{net} = \frac{1}{2}m(v_f^2 - v_i^2)$$

$$W_{net} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

$$W_{net} = K.E_f - K.E_i = \Delta K.E$$

This is the expression of kinetic energy for the object under the action of non-constant force.

Power

The rate of doing work is called power. If an agent does work 'ΔW' in time 'Δt', then the average power is defined as the ration to total work done to the total time. It is described mathematically as:

$$P_{av} = \langle P \rangle = \frac{\Delta W}{\Delta t}$$

If the power is variable, then the instantaneous power is given by the expression:

$$P_{ins} = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

Watt

The SI unit of power is watt which can be defined as:

"If an agent does work of one joule of work per second, the power of that agent will be 1 watt"

Question: Prove that $P = \vec{F}.\vec{v}$

Proof: As $P = \frac{dW}{dt}$

Here the work done $dW = \vec{F}.d\vec{s}$, therefore

$$P = \frac{\vec{F}.d\vec{s}}{dt} = \vec{F}.\frac{d\vec{s}}{dt}$$

But $\frac{d\vec{s}}{dt} = \vec{v}$

So $P = \vec{F}.\vec{v}$