PARTICLE DYNAMICS

Force Laws

There are four fundamental forces in nature.

Gravitational force

This force originates due to the presence of matter.

Electromagnetic force

This force includes basic electric and magnetic interactions and is responsible for the binding of atoms and the structure of solids.

Weak Nuclear Force

This force causes certain radioactive decay processes and certain reactions among the fundamental particles.

Strong Force

This force operates among the fundamental particles (protons and neutrons) and is responsible for biding the nucleus together.

Example:

For the case of two protons, the forces have the following relative strengths:

Strong Force:	Relative Strength = 1
Electromagnetic Force:	Relative Strength = 10^{-2}
Weak Nuclear Force:	Relative Strength = 10^{-7}
Gravitational Force:	Relative Strength = 10^{-38}

It is clear that the gravitational force is very weak and has negligible effects.

Electroweak Force

In 1967, a theory was proposed according to which weak and electromagnetic forces could be regarded as parts of a single force called electroweak force.

Grand Unification Theories

There are new theories proposed for the combination of strong and electroweak forces into a single force into a single framework.

Theories of Everything

The theories, which are proposed of the unification of all the four fundamental forces, are called theories of everything.

There are some other forces for which the electromagnetic force is the origin. For example contact forces such as normal force, frictional force, viscous force, tensile force, elastic force and many others. Microscopically, all these forces originate with the force between the atoms.

Frictional Forces

When one force moves against the other, then force is produced between the surfaces. This force opposes the motion of the bodies. For example if a block of mass m is projected with initial velocity v_0 along a horizontal table, then it will finally comes to rest. This is due to the fact that there is force of friction between the block and the surface of table which produces a negative acceleration a. this reduced the velocity of the block to zero.

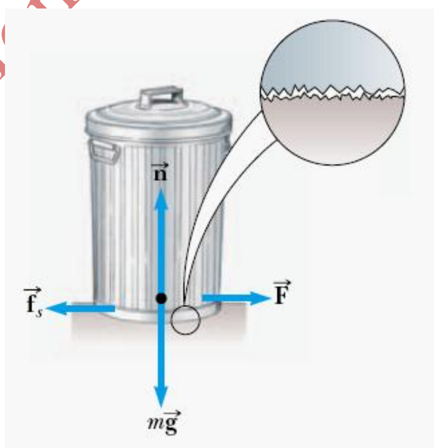
When one body slides on the surface of the other, then each body exerts a force of friction of the other surface. So, the frictional force opposes the relative motion. Even when there is no relative motion, frictional forces may exist between the surfaces.

In an automobile, about 20% of the engine power is used to counteract the frictional forces. Friction also causes wear and seizing of moving parts, therefore a lot of effort is made to reduce the friction. But on the other hand, friction is very important in our daily life, because it brings every rotating shaft to a halt. Without friction, we cannot walk, we cannot hold a pencil and could not write. Also, wheeled transportation is only possible due to friction.

Force of Static Friction

The frictional forces acting between the surfaces at rest with respect to each other are called forces of static friction.

Consider a block of mass m placed on a horizontal surface. The weight of block is mg, which is balanced by the normal force N (reaction of the horizontal surface) as shown in the figure.



Suppose a force F is applied on the resting block which is balanced by the equal and opposite force of static friction f_S. As F increases, the force of static friction also increases. Until f_S reaches a certain maximum value just before sliding the block. This force of static friction depends upon

The normal force N

The nature of the surfaces in contact

$$\mathbf{f}_S \le \boldsymbol{\mu}_S \boldsymbol{N}$$

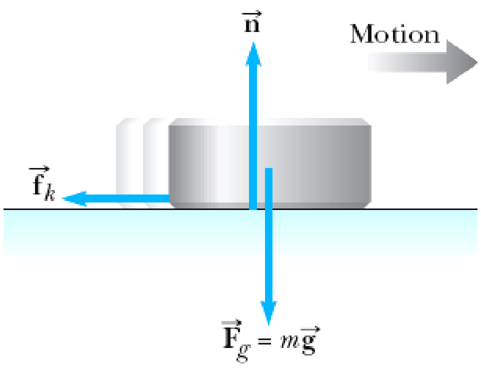
Or $\quad (f_S)_{max} = \mu_S N$

Here $(f_S)_{max}$ is the maximum value of the force of static friction, just before the sliding or moving of the block. μ_S is called the coefficient of static friction. It depends upon the nature of the surfaces in contact.

Force of Kinetic Friction

When the value of applied force F is greater than the maximum force of static friction $(f_S)_{max}$, then the block starts moving and has accelerated motion, i.e.,

$F > (f_S)_{max}$ Let the value of the applied force F is so adjusted that the block moves with uniform velocity v. in this case a



force of friction is also present which is called force of kinetic friction f_K. This force also depends upon:

- The normal force
- The nature of the surfaces sliding against each other

When the block is moving with uniform velocity,

$$f_K = \mu_K N$$

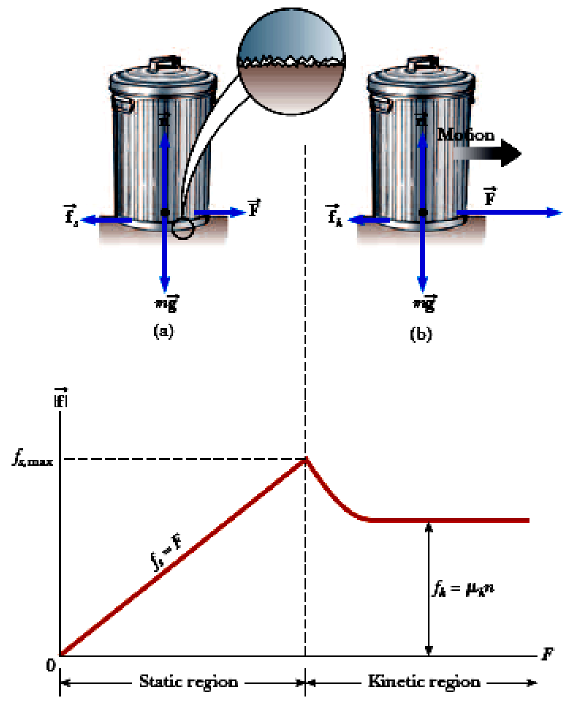
Where μ_K is called the coefficient of kinetic friction.

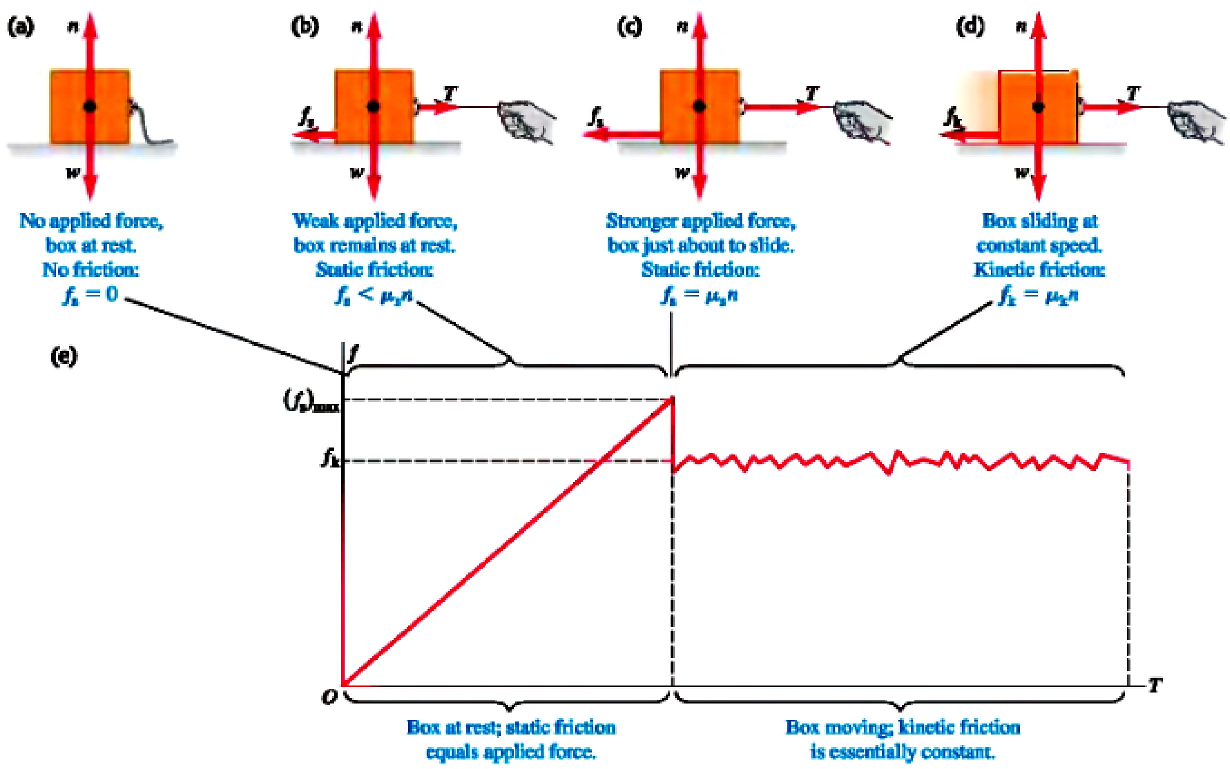
It should be noted that

$$(f_S)_{max} > f_K$$

Also,

$$\mu_f > \mu_K$$



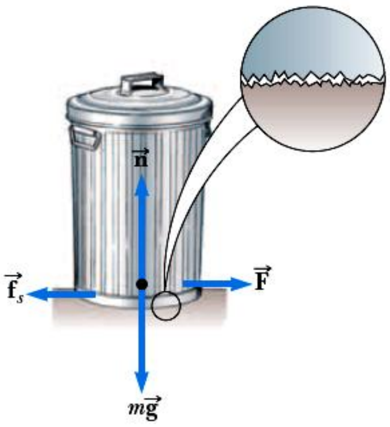


The Microscopic Basis of Friction

On the atomic scale, even the most finely polished surface is far from plane. For example, a highly polished steel surface has irregularities. The surface irregularities is several thousands atomic diameters.

When the bodies are placed incontact, then the actual microscopic area of contact is much less than the true area of the surface. In a particular case, these areas can easiliy be in the ratio 1:10000 . the actual microscopic area of contact is proportional to the mormal force, because the contact points deform plastically under the great stresses that develop at these points. Therefore, many contact points actually becomes cold welded together.

This phenomenon of "surface adhesion" occurs because at the contact points, the molecules on the opposite side of the surfaces are so close together that they exert strong intermolecular forces on each other.



The coefficient of friction depends upon many variables such as:

- The nature of surface of materials
- Surface finish
- Surface films
- Temperature

In the absence of air, oxide films may form on the opposite surface, which reduce the friction.

The Dynamics of Uniform Circular Motion

Consider of body of mass m, which is moving with uniform speed v along a circular path of radius r. as the direction of the body changes continuously, therefore, it has variable velocity and it has some acceleration a, which is directed radially inward i.e., towards the center of the circle. This is called centripetal acceleration and is given by:

$$a = \frac{v^2}{r}$$

Hence $\bar{a}$ is a variable vector because even though its magnitude remains constant, its direction changes continuously.

The net force acting upon the body is called centripetal force, which is given by the Newton's second law of motion:

$$\sum \bar{F} = m\bar{a}$$

$$\left|\sum \bar{F}\right| = ma = m\frac{v^2}{r}$$

The body moving in the circle is not in equilibrium state, because the net force acting the body is not zero.

Centripetal Force

The force which is responsible for uniform circular motion and is always directed towards the center of the circle is called centripetal force.

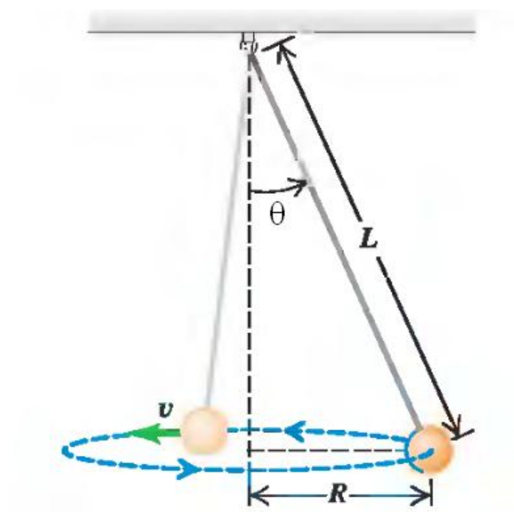
Consider a disc of mass m on the end of a string and is moving with constant speed v along circular path of radius R. In this case, the centripetal force applied on the disc is the tension T in the string.

As the moon is revolving around the earth, therefore, the centripetal force is the gravitational pull of the earth. Similarly, for the case of the electrons revolving around the nucleus, the centripetal force is provided by the force of attraction.

The Conical Pendulum

Figure shows a small body of mass m revolving in a horizontal circle of radius R with constant speed v at the end of a string of length L. as the body swings around, the string sweeps over the surface of an imaginary cone. This device is called conical pendulum.

The string describes a right circular cone of semi-angle θ. Let T is the tension in the string and the weight mg of the conical pendulum is acting vertically downward. The string makes and angle θ with vertical. According to Newton's second law of motion, the net force acting on the conical pendulum is:



$$\sum \overline{F} = T + m\overline{g} = m\overline{a}$$

The tension in the string can be resolved into two rectangular components:

i. The redial component T_r directed towards the center of the circle

$$T_r = -T\sin\theta$$

ii. The vertical component T_z directed upwards

$$T_z = T\cos\theta$$

As there is no vertical acceleration, therefore, vertical forces are balanced:

$$\sum \overline{F}_z = T\cos\theta - m\overline{g} = 0$$

$$T\cos\theta = m\overline{g} \qquad \text{----------} \quad (1)$$

The centripetal force acting on the conical pendulum is equal to the radial component of tension:

$$\sum \overline{F}_r = T_r = ma_r \qquad \text{----------} \quad (2)$$

But $T_r = -T\sin\theta$

And $a_r = -\dfrac{v^2}{R}$

The equation (2) will become:

$$-T\sin\theta = -m\frac{v^2}{R}$$

$$T\sin\theta = m\frac{v^2}{R} \qquad \text{----------} \quad (3)$$

Dividing equation (1) and (3), we get:

$$\frac{T\sin\theta}{T\cos\theta} = \frac{mv^2/R}{mg}$$

$$\tan\theta = \frac{v^2}{Rg}$$

$$v^2 = Rg\tan\theta$$

$$v = \sqrt{Rg\tan\theta}$$

This gives the constant speed of the conical pendulum.

Period of Motion

If t is the time for one revolution, then

$$t = \frac{2\pi R}{v} = \frac{2\pi R}{\sqrt{Rg\tan\theta}} = 2\pi\sqrt{\frac{R\cos\theta}{g\sin\theta}}$$

But $R = L\sin\theta$

$$t = 2\pi\sqrt{\frac{L\sin\theta.\cos\theta}{g\sin\theta}}$$

$$t = 2\pi\sqrt{\frac{L\cos\theta}{g}}$$

The Rotor

The rotor is a hollow cylindrical room. A person enters the rotor, closes the door and stands against the wall. The rotor starts rotating about vertical axis. When it got sufficient speed (pre-determined) then the floor below the person is opened downward, revealing a deep pit. The person does not fall but remains pinned up against the wall of the rotor.

Here N is the normal force exerted by the wall on the person, which is also equal to the centripetal force acting on the person in the rotor.

If the person does not fall then there must be no acceleration along vertical direction. The weight must be balanced by the force of static friction:

$$a_z = 0$$

$$\sum F_z = f_s - mg = ma_z = 0$$

$$\Rightarrow f_s - mg = 0$$

$$\Rightarrow f_s = mg \quad \text{----------} \quad (1)$$

Let R is the radius of the rotor and v is the speed with which the person isrotating alosn the circular path, then the centripetal or radial acceleration is:

$$a_r = -\frac{v^2}{R}$$

Now along the radial direction, the sum of component of forces is:

$$\sum F_r = -N$$

$$\Rightarrow ma_r = -N$$

$$\Rightarrow m\left(-\frac{v^2}{R}\right) = -N$$

$$\Rightarrow -m\frac{v^2}{R} = -N$$

Written and composed by: Prof. Muhammad Ali Malik (M. Phil. Physics), Govt. Degree College, Naushera

$$\Rightarrow \frac{mv^2}{R} = N \quad ---------- \quad (2)$$

The force of static friction is:

$$f_s = \mu_s N$$

$$f_s = \mu_s \left(\frac{mv^2}{R} \right)$$

here μ_s is the coefficient of static friction. Putting values in equation (1), we get:

$$\frac{\mu_s mv^2}{R} = mg$$

$$v = \sqrt{\frac{Rg}{\mu_s}}$$

This is the expression f velocity of the rotor, beyond which the person in the rotor does not fall and remains pinned against the wall of the rotor.

Equations of Motion under Constant Force

A constant force produces constant acceleration and the acceleration is described as the derivative of velocity. If a and v is the acceleration and velocity of the moving object, then acceleration of the object is described as:

$$a = \frac{dv}{dt}$$

Or $dv = a\,dt$

Integrating the above equation between the limits:

At $t=0$, velocity = v_0

At $t=t$, velocity = v

$$\int_{v_0}^{v} dv = a \int_{0}^{t} dt$$

As the acceleration is constant:

$$|v|_{v_0}^{v} = a\,|t|_0^t$$

$$v - v_0 = at$$

$$v = v_0 + at$$

The time derivative of position vector x is equal to the velocity v, i.e.,

$$v = \frac{dx}{dt}$$

Or $dx = v\,dt$

Putting the value of v from equation (1), we have:

$$dx = (v_0 + at)\,dt$$

Integrating the above equation between the limits:

At $t=0$, Position w r t origin = x_0

At $t = t$, velocity = v

$$\int_{x_0}^{x} dx = \int_{0}^{t} ((v_0 + at))dt$$

$$\int_{x_0}^{x} dx = \int_{0}^{t} v_0 \, dt + \int_{0}^{t} at \, dt$$

$$|x|_{x_0}^{x} = v_0 |t|_0^t + a\left|\frac{t^2}{2}\right|_0^t$$

$$x - x_0 = v_0 t + a\frac{t^2}{2}$$

$$x = x_0 + v_0 t + a\frac{t^2}{2}$$

Example of Non-Constant Forces

There are some forces which are not constant but these forces change with respect to time, velocity or position.

Forces Depending on Time

To stop a moving car, brakes are applied slowly at first and then more strongly as the car slows. In this case, the braking force depends on the time during the interval when the car is slowing. Another example of time dependent force is that force which is applied by the sound waves on the air molecules during their propagation. As the sound waves vary sinusoidally with respect to time, therefore, the forces also change sinusoidally with respect to time.

Force Depending on Velocity

When a body is moving through a fluid medium, such as air or water, then the frictional force or drag force acting upon the body increases with increase in velocity of the body.

In case of free fall, the drag force increases up to the limit that it balances the weight of the body and then the body falls with constant velocity, known as terminal velocity. To approach the limit of terminal velocity, the free fall must be of the order of 100 m or so. Similarly the projectile motion is also affected severely by the drag force due to which the range can be reduced on half or less.

If we walk slowly in a swimming pool, we feel only a small resistive force. But it we try to walk quicky, the resistive force will also increase.

Forces Depending upon the Position

The restoring force applied by a spring on a body of mass m is the example of the force which depends on position. The resorting force F is directly proportional to the displacement x, of the body from mean position:

$$F \ \alpha \ -x$$

$$F = -kx$$

Using Newton's 2nd law of motion:

$$ma = -kx$$

Where m is mass of the object and a is the acceleration with which the object is moving.

$$a = -\frac{k}{m}x$$

The restoring force will be zero at mean position and it will become maximum at extreme position. In a similar way, the acceleration also increases and decreases as the position of the vibrating body changes.

Time Dependent Forces (Analytical Method)

Let a(t) is the time dependent acceleration due to a time dependent force. Then in is given by the expression:

$$a(t) = \frac{dv}{dt}$$

Or $dv = a(t)\ dt$

Integrating the above equation between the limits:

- At $t=0$, velocity = v_0
- At $t=t$, velocity = v

$$\int_{v_0}^{v} dv = \int_0^t a(t)\,dt$$

$$|v|_{v_0}^{v} = \int_0^t a(t)\,dt$$

$$v - v_0 = \int_0^t a(t)\,dt$$

$$v = v_0 + \int_0^t a(t)\,dt$$

Once we have $v(t)$, we can calculate $x(t)$

The time derivative of position vector x is equal to the velocity v, i.e.,

$$v(t) = \frac{dx(t)}{dt}$$

Or $dx = v(t)\,dt$

Integrating the above equation between the limits:

- At $t=0$, Position w r t origin = x_0
- At $t=t$, velocity = v

$$\int_{x_0}^{x} dx = \int_0^t v(t)\,dt$$

$$|x|_{x_0}^{x} = \int_0^t v(t)\,dt$$

$$x - x_0 = \int_0^t v(t)\,dt$$

$$x = x_0 + \int_0^t v(t)\,dt$$

Drag Forces and the Motion of Projectile

Raindrops fall from the clouds whose height h above the ground is about 2 km. the expected velocity of the raindrop on striking the ground is $v = \sqrt{2gh} \cong 200\,ms^{-1}$. But the actual velocity of the raindrop is much smaller. This is due to the drag force i.e., frictional force of air on the raindrop.

The drag force acting on an object depends upon its velocity. Greater the velocity, greater is the drag force. The velocity of the object can increase to a constant value which is known as terminal speed. In this case, the force and acceleration is velocity dependent.

$$a(v) = \frac{dv}{dt}$$

$$dt = \frac{dv}{a(v)}$$

Integrating the above equation between the limits:

- At $t = 0$, velocity = v_0
- At $t = t$, velocity = v

$$\int_0^t dt = \int_{v_0}^{v} \frac{dv}{a(v)}$$

$$\Rightarrow t = \int_{v_0}^{v} \frac{dv}{a(v)}$$

SAMPLE PROBLEM

Consider an object of mass m falling in air experiences a draf force D, which increases linearly with velocity:

$$D \,\alpha\, v$$

$$D = bv$$

Here b is the constant depending on the properties of the object (its size and shape) and also on the properties of the fluid (especially its density). We have to find the velocity as a function of time, $v(t)$, when the mass is dropped from rest $v_0 = 0$.

Solution

The net force acting on the object in the downward direction is

$$\sum F_y = mg - bv$$

$$ma = mg - bv$$

$$a = g - \frac{b}{m}v \quad \text{----------} \quad (1)$$

For the case of a velocity dependent force, we can write:

$$a(v) = \frac{dv}{dt}$$

$$dt = \frac{dv}{a(v)}$$

Integrating the above equation between the limits:

- At $t = 0$, velocity = v_0
- At $t = t$, velocity = v

$$\int_0^t dt = \int_{v_0}^{v} \frac{dv}{a(v)}$$

$$\Rightarrow t = \int_{v_0}^{v} \frac{dv}{a(v)} \quad \text{----------} \quad (2)$$

Putting values from equation (1), we have:

$$\Rightarrow t = \int_{v_0}^{v} \frac{dv}{\left(g - \frac{b}{m}v\right)}$$

For the present case, $v_0 = 0$, therefore

$$t = \int_0^v \frac{dv}{\left(g - \frac{b}{m}v\right)}$$

$$t = m\int_0^v \frac{dv}{(mg - bv)}$$

$$t = -\frac{m}{b}\int_0^v \frac{-bdv}{(mg - bv)}$$

$$t = -\frac{m}{b}\left|\ln(mg - bv)\right|_0^v$$

$$t = -\frac{m}{b}\left[\ln(mg - bv) - \ln(mg)\right]$$

$$t = -\frac{m}{b}\ln\left(\frac{mg - bv}{mg}\right)$$

$$\ln\left(\frac{mg - bv}{mg}\right) = -\frac{bt}{m}$$

$$\left(\frac{mg - bv}{mg}\right) = e^{\left(-\frac{bt}{m}\right)}$$

$$mg - bv = mg\, e^{\left(-\frac{bt}{m}\right)}$$

$$bv = mg - mg\, e^{\left(-\frac{bt}{m}\right)}$$

$$bv = mg\left(1 - e^{\left(-\frac{bt}{m}\right)}\right)$$

Written and composed by: Prof. Muhammad Ali Malik (M. Phil. Physics), Govt. Degree College, Naushera

$$v = \frac{mg}{b}\left(1 - e^{\left(-\frac{bt}{m}\right)}\right) \quad \text{----------} \quad (3)$$

Case 1. When t is small:

When t is small, then by using formula $e^x = 1 + x + \frac{x^2}{2!} + ...$ for $x < 1$ in equation (3), we have:

$$v = \frac{mg}{b}\left(\frac{bt}{m}\right) = gt$$

It means that for short interval of time, the object is falling freely under the action of gravity. The effect of drag force will be negligible for this case.

Case 2. When t is large:

When t is large, then

$$e^{\left(-\frac{bt}{m}\right)} \cong 0$$

The equation (3) will become:

$$v = \frac{mg}{b}$$

This velocity is known as terminal velocity.

Exercise Problem 59: An object is dropped from from rest. Find the terminal speed assuming that the drag force is given by $D = bv^2$

Solution. Assuming Newton's 2nd law:

$$F_{net} = mg - D = mg - bv^2$$

$$m\frac{dv}{dt} = mg - bv^2$$

$$\frac{dv}{dt} = g - \frac{bv^2}{m}$$

$$\frac{dv}{\left(g - \frac{bv^2}{m}\right)} = dt$$

$$\frac{dv}{\frac{b}{m}\left(\frac{mg}{b} - v^2\right)} = dt$$

$$\sqrt{\frac{mg}{b}} = v_T$$

$$\frac{dv}{\left(v_T{}^2 - v^2\right)} = \frac{b}{m}dt$$

Integrating both sides:

$$\int \frac{dv}{\left(v_T{}^2 - v^2\right)} = \int \frac{b}{m}dt$$

$$\frac{1}{2v_T}\ln\left|\frac{v_T+v}{v_T-v}\right|_0^v=\frac{b}{m}\left|t\right|_0^t$$

$$\frac{1}{2v_T}\left[\ln\left(\frac{v_T+v}{v_T-v}\right)-\ln\left(\frac{v_T+0}{v_T-0}\right)\right]=\frac{b}{m}t$$

$$\frac{1}{2v_T}\left[\ln\left(\frac{v_T+v}{v_T-v}\right)-\ln\left(1\right)\right]=\frac{b}{m}t$$

$$\ln(1)=0$$

$$\left[\ln\left(\frac{v_T+v}{v_T-v}\right)-0\right]=\frac{2v_Tb}{m}t$$

$$\ln\left(\frac{v_T+v}{v_T-v}\right)=\frac{2v_Tb}{m}t\Rightarrow\frac{v_T+v}{v_T-v}=\exp\left(\frac{2v_Tb}{m}t\right)$$

Applying Componendo-Dividendo Rule:

$$\frac{v_T+v+v_T-v}{v_T+v-v_T+v}=\frac{\exp\left(\frac{2v_Tb}{m}t\right)+1}{\exp\left(\frac{2v_Tb}{m}t\right)-1}\Rightarrow\frac{2v_T}{2v}=\frac{\exp\left(\frac{2v_Tb}{m}t\right)+1}{\exp\left(\frac{2v_Tb}{m}t\right)-1}\Rightarrow\frac{v_T}{v}=\frac{\exp\left(\frac{2v_Tb}{m}t\right)+1}{\exp\left(\frac{2v_Tb}{m}t\right)-1}$$

$$\Rightarrow\frac{v}{v_T}=\frac{\exp\left(\frac{2v_Tb}{m}t\right)-1}{\exp\left(\frac{2v_Tb}{m}t\right)+1}$$

$$v=v_T\left[\frac{\exp\left(\frac{2v_Tb}{m}t\right)-1}{\exp\left(\frac{2v_Tb}{m}t\right)+1}\right]$$

$$v=v_T\left[\frac{\exp\left(\frac{2v_Tb}{m}t\right)\left\{1-\exp\left(-\frac{2v_Tb}{m}t\right)\right\}}{\exp\left(\frac{2v_Tb}{m}t\right)\left\{1+\exp\left(-\frac{2v_Tb}{m}t\right)\right\}}\right]$$

$$v=v_T\left[\frac{\left\{1-\exp\left(-\frac{2v_Tb}{m}t\right)\right\}}{\left\{1+\exp\left(-\frac{2v_Tb}{m}t\right)\right\}}\right]$$

Terminal Speed:

For terminal Speed, put $t\rightarrow\infty$:

When $t\rightarrow\infty$, $\exp\left(-\frac{2v_Tb}{m}t\right)\rightarrow 0$

$$v=v_T\left[\frac{1-0}{1+0}\right]=v_T$$

Projectile Motion with Air Resistance

The two dimensional projectile motion is also affected by the drag force due to resistance of air. The height as well as the range of projectile is reduced. For example if a base ball is projected with initial velocity of 45 ms^{-1} at an angel of 60^0 with the horizontal, then its range is reduced from 179 m to 72 m and the maximum height is reduced from 78 m to 48 m. also the trajectory is no symmetric about the maximum the descending motion is much steeper than ascending motion. The projectile strikes the ground at an angel of - -79^0. The drag force depends upon the velocity of projectile. If the wind is blowing, the calculation must be changed accordingly and results will differ.

Projectile Motion

When a body is projected at an angel with the horizontal and it moves freely under the action of gravity is called a projectile. Projectile motion is an example of two dimensional motion in which the objects moves with constant acceleration.

Suppose a projectile of mass m is projected at an angle θ_0 with the horizontal with initial velocity v_0 and it moves in xy-plane. Let $\vec{r}$ r is the position vector and $\vec{v}$ be its velocity at any time, then according to second law of motion:

$$m\vec{a} = \vec{F} = -m\vec{g}\,\hat{j}$$

$$m\frac{d^2\vec{r}}{dx^2} = m\frac{d\vec{v}}{dx} = -m\vec{g}\,\hat{j}$$

$$\Rightarrow \frac{d\vec{v}}{dx} = -\vec{g}\,\hat{j}$$

Integrating both sides with respect to time, we get:

$$\int \frac{d\vec{v}}{dx}dt = -\int \vec{g}\,\hat{j}\,dt$$

$$\vec{v} = -\vec{g}\,\hat{j}\,t + A_1$$

Initially, at $t=0$, $\vec{v}=\vec{v}_0$, so $A_1 = \vec{v}_0$

$$\vec{v} = \vec{v}_0 - \vec{g}\,\hat{j}\,t \quad \text{----------} \quad (1)$$

As the motion of the object is in two dimensions, so the equation (1) can be written in terms of rectangular components as:

$$(v_x\hat{i} + v_y\hat{j}) = (v_{0x}\hat{i} + v_{0y}\hat{j}) - \vec{g}\,\hat{j}\,t$$

Or $$v_x\hat{i} + v_y\hat{j} = v_{0x}\hat{i} + \left(v_{0y} - \vec{g}t\right)\hat{j}$$

Comparing coefficients of $\hat{i}$ and $\hat{j}$ on both sides:

$$v_x = v_{0x} \qquad Or \qquad v_x = v_0\cos\theta_0$$

$$v_y = v_{0y} - \vec{g}t \qquad Or \qquad v_y = v_0\sin\theta_0 - gt$$

The equation (1) can be written as:

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{v}_0 - \vec{g}\,\hat{j}\,t$$

Integrating both sides, we get:

$$\int \frac{d\vec{r}}{dt}dt = \int \left(\vec{v}_0 - \vec{g}\,\hat{j}\,t\right)dt$$

$$\vec{r} = \vec{v}_0 t - \frac{1}{2}\vec{g}t^2\,\hat{j} + A_2$$

Initially, at $t=0$, $\vec{r}=0$, so $A_2 = 0$

Thus

$$\vec{r} = \vec{v}_0 t - \frac{1}{2}\vec{g}t^2\,\hat{j} \qquad \text{----------} \quad (2)$$

As the motion of the object is in two dimensions, so the equation (2) can be written in terms of rectangular components as:

$$(x\hat{i} + y\hat{j}) = (v_{0x}\hat{i} + v_{0y}\hat{j})t - \frac{1}{2}\vec{g}t^2\,\hat{j}$$

Or $$x\hat{i} + y\hat{j} = v_{0x}t\,\hat{i} + \left(v_{0y}t - \frac{1}{2}\vec{g}t^2\right)\hat{j}$$

Comparing coefficients of $\hat{i}$ and $\hat{j}$ on both sides, we get:

$$x = v_{0x}t \qquad or \qquad x = v_0 \cos\theta_0 t$$

$$y = v_{0y}t - \frac{1}{2}\vec{g}t^2 \qquad or \qquad y = v_0 \sin\theta_0 t - \frac{1}{2}\vec{g}t^2$$

Trajectory of the Projectile

As $x = v_0 \cos\theta_0 t \Rightarrow t = \dfrac{x}{v_0 \cos\theta_0}$

Putting this value of t in

$$y = v_0 \sin\theta_0 t - \frac{1}{2}gt^2$$

$$y = v_0 \sin\theta_0 \left(\frac{x}{v_0 \cos\theta_0}\right) - \frac{1}{2}g\left(\frac{x}{v_0 \cos\theta_0}\right)^2$$

$$y = x\tan\theta_0 - \frac{1}{2}g\frac{x^2}{v_0^2\cos^2\theta_0}$$

We can write above equation as:

$$y = ax - bx^2 \qquad \text{----------} \quad (3)$$

Where $a = \tan\theta_0$, *and* $b = \dfrac{1}{2}\dfrac{g}{v_0^2\cos^2\theta_0}$

Equation (3) is the equation of parabola. So the trajectory of projectile is parabola.

Magnitude of Velocity at any Instant

The magnitude of velocity can be find out by using formula:

$$v = \sqrt{v_x^2 + v_y^2}$$

$$v = \sqrt{(v_0\cos\theta_0)^2 + (v_0\sin\theta_0 - gt)^2}$$

$$v = \sqrt{v_0^2\cos^2\theta_0 + v_0^2\sin^2\theta_0 + g^2t^2 - 2v_0gt\sin^2\theta_0}$$

$$v=\sqrt{v_0^{\,2}\left(\cos^2\theta_0+\sin^2\theta_0\right)+g^2t^2-2v_0gt\sin^2\theta_0}$$

$$v=\sqrt{v_0^{\,2}+g^2t^2-2v_0gt\sin^2\theta_0}$$

Direction of Velocity at any Instant

The angle θ which the velocity makes, at instant, with horizontal can be find out by using expression:

$$\tan\theta=\frac{V_y}{V_x}=\frac{v_0\sin\theta_0 - gt}{v_0\cos\theta_0}$$

Time to Reach at Maximum Height (tm)

As the vertical component of velocity is

$$v_y = v_0\sin\theta_0 - gt$$

At highest point $v_y = 0$, therefore,

$$0 = v_0\sin\theta_0 - gt$$

$$gt = v_0\sin\theta_0$$

$$t_m=\frac{v_0\sin\theta_0}{g}$$

Time of Flight (T)

It is the time taken by the projectile from the point of projection to come back to the level of projection.

$$y = v_0\sin\theta_0 t - \frac{1}{2}gt^2$$

As vertical displacement of the projectile $y=0$, so

$$0 = v_0\sin\theta_0 t - \frac{1}{2}gt^2$$

$$\frac{1}{2}gt^2 = v_0\sin\theta_0 t$$

$$t=\frac{2v_0\sin\theta_0}{g}$$

Here t is the time of flight, i.e., t =T:

$$T=\frac{2v_0\sin\theta_0}{g}$$

Horizontal Range

It is the horizontal distance covered by the projectile. As the horizontal component of velocity for a projectile remains constant $(a_x=0)$, so by using the 2nd equation of motion:

$$R=v_x\times T$$

Where $v_x\left(=v_0\cos\theta_0\right)$ is the horizontal component of velocity and $T\left(=\frac{2v_0\sin\theta_0}{g}\right)$ is the time of flight:

$$R=v_0\cos\theta_0\times\frac{2v_0\sin\theta_0}{g}$$

$$R = \frac{2v_0^{\ 2}\sin\theta_0\cos\theta_0}{g}$$

$$R = \frac{v_0^{\ 2}\left(2\sin\theta_0\cos\theta_0\right)}{g}$$

$$R = \frac{v_0^{\ 2}\sin 2\theta_0}{g}$$

Maximum Range

The range of the projectile will be maximum, when

$$\sin 2\theta_0 = 1$$

$$\Rightarrow 2\theta_0 = \sin^{-1}(1)$$

$$\Rightarrow 2\theta_0 = 90^0$$

$$\Rightarrow \theta_0 = 45^0$$

Thus the projectile will have the maximum range when it will be projected at an angle of 45^0. Therefore:

$$R = \frac{v_0^{\ 2}}{g}$$

Non-Inertial Frames and Pseudo Forces

To apply the classical mechanics in non-inertial frames, we must introduce additional forces known as pseudo-forces. Unlike other forces, we can not associate pseudo-forces with any particular object in the environment of the body on which they act. Moreover, if we view the body from an inertial frame, the pseudo forces disappear. Pseudo forces are simply devices that permit us to apply classical mechanics in the normal way to events if we insist on viewing the events from a non-inertial reference frame.

Linearly Accelerated References Frames

Consider an observer S' riding in a van that is moving at constant velocity. The van contains a long air-track with a frictionless 0.25 glider resting at one end. The driver of the van applies the brakes, ant the van begins to decelerate. An observer S on the ground measures the constant acceleration of the van to be -2.8 ms^{-2}. The observer S' riding in the van is therefore in a non-inertial frame of reference when the van begins to decelerate. The observer S' observes the glider to move down the track with an acceleration of 2.8 ms^{-2}.

For ground observer S, who is an inertial frame of reference, the analysis is straight forward. The glider, which had been moving forward at constant velocity before the van started to brake, simply continues to do so. According to S, the glider has no acceleration and therefore no horizontal force need be acting on it.

Observer S', however, sees the glider accelerate and can find no object in the environment of the glider that exerted a force on it to to provide its observed forward acceleration. To preserve the applicability of Newton's second law, S' must assume that a pseudo force acts on glider. According to S', this force F' must equal ma', where $a'(=-a)$ is the acceleration of the glider measured by observer S'. The magnitude of this pseudo force is:

$$F' = ma' = (0.25\,kg)(2.8\,ms^{-2}) = 0.70\,N$$

And its direction is the same as a', that is towards the front of van.

This force which is very real from the point of view of S', is no apparent to the ground observer S.

Driving a Car on Circular Path

Pseudo forces are very real to those that experience them. Imagine yourself riding in a car that is rounding a curve to the left. To a ground observer, the car is experiencing a centripetal acceleration and therefore constitutes a non-inertial reference frame. To the ground observer, who is in inertial frame of reference, this is quite natural: your body is simply trying to obey the Newton's first law and moves in a straight line. From your point of view in non-inertial frame of reference of car, you must ascribe your sliding motion to a pseudo-force pulling you to the right. This type of pseudo force is called centrifugal force meaning a force directed away from center.

Centrifuge Machine

Pseudo forces can be used as the basis of practical devices. Consider the centrifuge, one of the most useful of laboratory instruments. As a mixture of substances moves rapidly in a circle, the more massive substances experience a larger centrifugal force $\frac{mv^2}{r}$ and move further away from the axis of rotation. The centrifuge, thus uses a pseudo-force to separate substance by mass, just as mass spectrometer uses electromagnetic force to separate atoms by mass.

Limitations of Newton's Laws

In 20^{th} century, the physical world has experienced three revolutionary developments:

- Einstein's Special Theory of Relativity (1905)
- Einstein's General Theory of Relativity (1915)
- Quantum Mechanics (1925)

Special theory of relativity teaches that we can't extrapolate the use of Newton's laws to particles moving at speed comparable to the speed of light. General theory of relativity shows that we can't use Newton's laws in the vicinity of very large gravitational force. Quantum mechanics teaches us that we can't extrapolate the Newton's laws to the objects as small as atom.

Special theory, which involves a distinctly non-Newtonian view of space and time, can be applied under all circumstances, at high speed and low speeds. In the limit of low speed, it can be shown that the dynamics of special reduces directly to the Newton's laws.

Similarly, general theory can be applied to weak as well as strong gravitational fields, but its equation reduces to Newton's laws for weak forces.

Quantum mechanics can be applied to the individual atoms, where certain randomness in behavior is predicted. To ordinary objects containing huge number of atoms, the randomness averages out to give Newton's laws once again.